THAI NGUYEN UNIVERSITY
OF TECHNOLOGY

AWCN Lab

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Text Book

SIGNAL ANALYSIS

SCIENCE AND TECHNICS PUBLISHING HOUSE

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CONTENTS

PREFACE

Signal analysis is a subject that arises in many different disciplines such as science, engineering, and economics. Scientists and engineers all use the concept of signals and systems because the concept forms a foundation on which we build many things of our daily lives. Typical examples of systems include radio and television, telephone networks, radar systems, computer networks, wireless communication, military surveillance systems, and satellite communication systems. And we interact with those systems via signals.

A signal typically delivers information about the nature of a physical phenomenon. Examples of signals include atmospheric temperature, humidity, human voice, and television images. Necessary information within a signal, however, does not come to our hand for free. It requires an art of signal analysis. Moreover, most engineering students will deal with signals and systems in their professional lives in future. That is why a course on signal analysis is an important part of engineering curricula, and we cannot overemphasize the importance of understanding signal analysis.

Unfortunately, however, engineering curricula are so crowded, and many students cannot take enough credit for signal analysis. It is, in fact, not hard to see students studying signal analysis only for one semester and finishing the study in the middle of ongoing discussion about the signal analysis. This discrepancy between the importance of signal analysis and the curricular reality necessitates us to write this book.

This book is, in principle, written for sophomore or junior level students, who encounter the topic of signal analysis for the first time. And this book is organized in such a way that students may experience the essence of signal analysis within a semester. We do not aim to cover every detail about signal analysis. We do not try to write a book that deserves someone's lifelong reference. There are already a lot of classic books that deserve the honor. Instead of that, we hope this book to be an agent that helps students successfully plunge into the world of signal analysis.

Finally, we emphasize that mathematics is so much about studying signal analysis. The mathematical prerequisite for this book is standard mathematics that includes calculus and differential equations. Readers are, however, supposed to parallel studying complex numbers along with the study of signal analysis. Up to Chapter 6, we intentionally avoid expressing things in terms of complex numbers. By the start of Chapter 7, however, students should already have a good understanding about the theory of complex numbers.

LIST OF ABBREVIATIONS

AM	Amplitude Modulation
BIBO	Bounded-Input and Bounded-Output
BPF	Band Pass Filter
BSF	Band Stop Filter
CT Signal	Continuous-Time Signal
DC	Direct Current
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
DT Signal	Discrete-Time Signal
FFT	Fast Fourier Transform
FM	Frequency Modulation
HPF	High Pass Filter
LCM	Least Common Multiple
LPF	Low Pass Filter
LTI System	Linear Time-Invariant System
ROC	Region Of Convergence

Chapter 1

BASIC CONCEPTS

While discussing signal analysis, we often encounter the multi-dimensional nature of the subject. The first dimension is based on which element of a problem we focus on: signal or system. Another dimension is dependent on what form of data we deal with: continuous or discrete. Furthermore, we frequently move between two different domains: time or frequency. In other words, our discussion may start with continuous-time signals in frequency domain and then migrate to discrete-time systems in time domain. And, without clear understanding about what we are handling at a moment, we quickly get lost in the middle of discussing signal analysis. Therefore, we first clarify terminologies shown in Figure 1.1 and then introduce several basic signals.

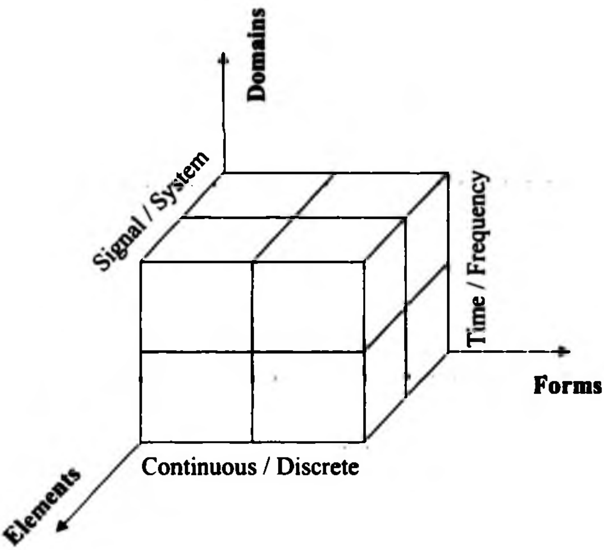


Figure 1.1: 2 elements, 2 forms, and 2 domains

1.1 SIGNAL AND SYSTEM

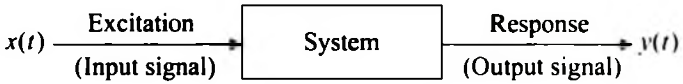


Figure 1.2: Signal and system

A *signal* is a set of data or function that represents a phenomenon of interest. Almost every measurable quantity can be the signal of our concern. It can be an acoustic recording with time ($A(t)$) or wind intensity measurement along a road ($W(x)$). There can be 2-dimensional (e.g. a picture image $P(x, y)$) or higher dimensional signal (e.g. a movie $M(x, y, t)$). In this study, we predominantly focus on 1-dimensional signals that vary with time.

A *system* is a collection of devices that operates on an input signal $x(t)$ and produces an output signal $y(t)$. For example, we may regard voltages or currents in an electric circuit as signals, while regarding the circuit itself as a system. Note that as long as we discuss 1-dimensional time signals, the two symbols x and y always denote input and output signals, respectively.

1.2 CONTINUOUS-TIME AND DISCRETE-TIME

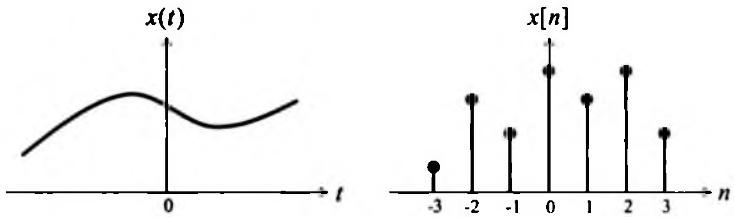


Figure 1.3: Continuous-time function $x(t)$ and discrete-time sequence $x[n]$

A *continuous-time signal* $x(t)$ is defined at every instant of time, while a *discrete-time signal* $x[n]$ is only defined at specific moments that are identified by the integer variable n. Throughout this study, continuous-time signals are also referred to as *time functions*, and discrete-time signals as *time sequences*.

One interacts with a *continuous-time system* via continuous-time signals and with a *discrete-time system* via discrete-time signals. Since time is naturally a continuous physical quantity, most physical systems are continuous-time systems. Discrete-time signals are usually obtained from continuous-time signals through *sampling*. There are, however, systems that are intrinsically discrete-time. Without a new deposit or withdrawal, the daily balance of a savings account remains the same for a day, and, thus, the savings account is a good example of discrete-time systems.

1.3 TIME DOMAIN AND FREQUENCY DOMAIN

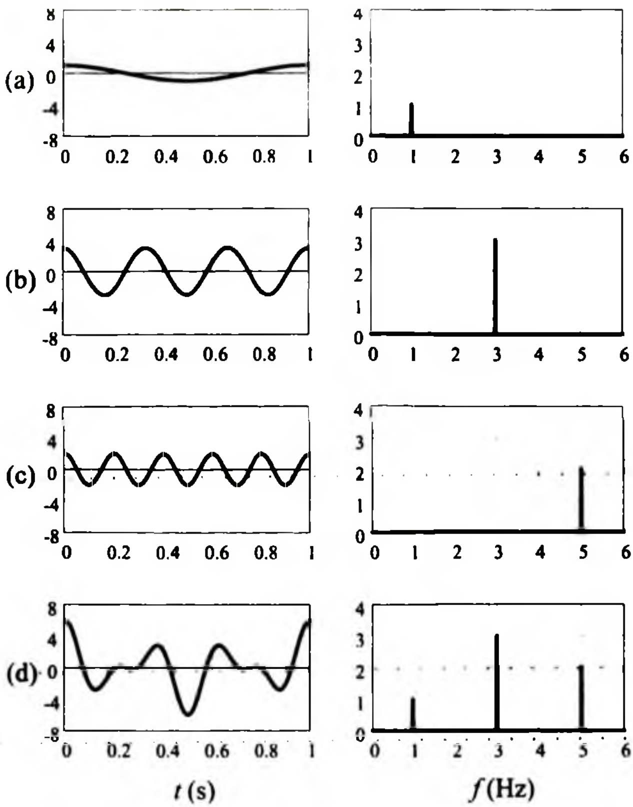


Figure 1.4: Time domain (left column) and frequency domain (right column) representations

Characteristics of a data set may be described in time or frequency domains. In the *time domain*, data characteristics are displayed along the time axis, and, in the *frequency domain*, on the other hand, along the frequency axis. Consider graphs on the left column of Figure 1.4, where continuous-time signals are displayed for 1 second. One can quickly read frequencies and amplitudes from the signals: frequency is 1 Hz and amplitude is 1 in row (a), frequency is 3 Hz and amplitude is 3 in row (b), and frequency is 5 Hz and amplitude is 2 in row (c). Graphs on the right column depict exactly the same information: frequencies by horizontal locations and amplitudes by vertical heights. One may find it is redundant to represent data in the frequency domain. However, in row (d), where signals of the upper three rows are summed up, it is evident that frequency information is strongly obscured in the time domain graphs, while well preserved in the frequency domain graphs.

Amplitude is, in fact, not the only information we present in the frequency domain. We will later study that phase is another crucial information we have to investigate in the frequency domain. Note that *amplitude spectrum* and *phase spectrum* mean that amplitude and phase information are being displayed along the frequency axis, respectively.

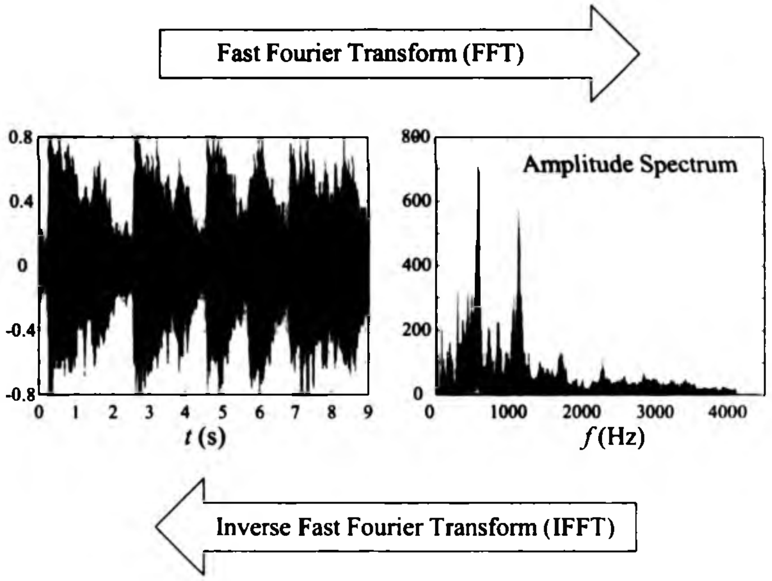
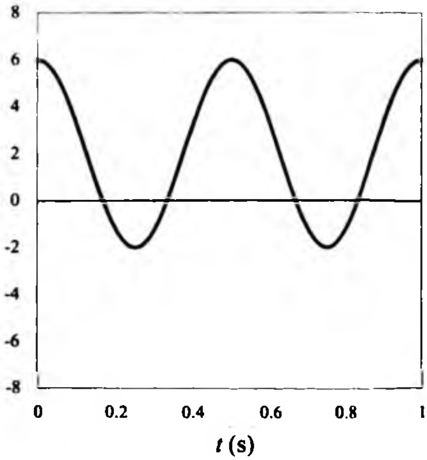


Figure 1.5: Fast Fourier transform and inverse fast Fourier transform

Figure 1.4 well validates the frequency domain analysis. We have not yet, however, discussed how to move between time and frequency domain. Consider, for example, the time signal shown in Figure 1.5. It is unreasonable to identify frequency characteristics of the time signal via any analytic means, and we need to rely on digital computer algorithms. The most popular algorithm for retrieving frequency information out of a time signal is called the *fast Fourier transform* (FFT). The topic about the FFT is covered later in Chapter 12.

Example 1.1 Sketch the amplitude spectrum of the time function shown below.

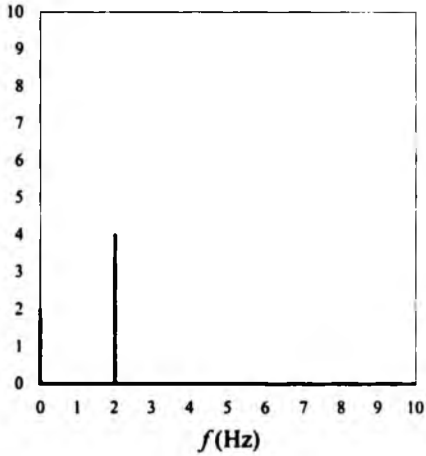


Solution

The signal fluctuates twice a second, and the range of fluctuation is between 6 and -2. The signal thus has a 2 Hz frequency component and a DC (0 Hz) component as well. And we identify the analytic expression of the function as

$$x(t) = 2 + 4\cos(4\pi t),$$

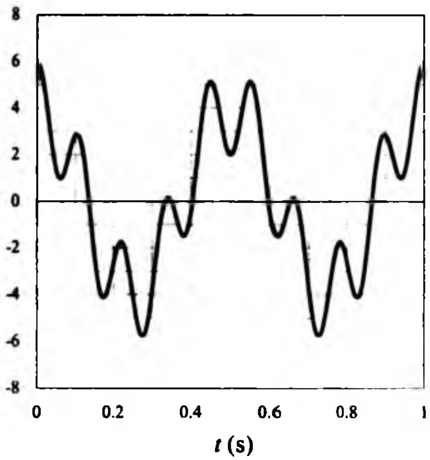
and sketch the amplitude spectrum as follows:



Example 1.2 Consider the time signal shown below. The signal has two frequency components:

$$x(t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t).$$

Sketch the amplitude spectrum of the time signal.



Solution

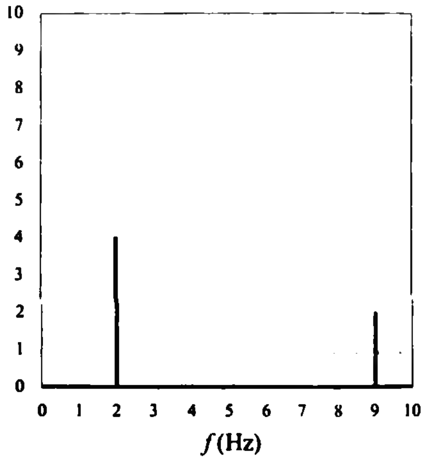
The signal exhibits low frequency fluctuation with 2 Hz frequency. The signal also shows high frequency variation with 9 Hz frequency. We thus write the expression of the signal as

$$x(t) = A_1 \cos(4\pi t) + A_2 \cos(18\pi t).$$

We also read the signal and write the following relations:

$$x(0.0) = A_1 + A_2 = 6,$$
$$x(0.5) = A_1 - A_2 = 2.$$

Solving the above expressions, we identify that $A_1 = 4$ and $A_2 = 2$, and we sketch the amplitude spectrum as follows:



1.4 BASIC CONTINUOUS-TIME SIGNALS

We present several basic continuous-time signals. These include the unit ramp function $r(t)$, unit step function $u(t)$, and unit impulse function $\delta(t)$. These three functions are also called *singularity functions*, because they are discontinuous or have discontinuous derivatives.

1.4.1 Unit Ramp Function

The *unit ramp function* $r(t)$ is defined as

$$r(t) = \begin{cases} t & (t \geq 0), \\ 0 & (t < 0). \end{cases} \tag{1.1}$$

Using the definition, we also write the following relations:

$$\begin{aligned} r(t+t_0) &= \begin{cases} t+t_0 & (t+t_0 \geq 0) \\ 0 & (t+t_0 < 0), \end{cases} \\ r(t-t_0) &= \begin{cases} t-t_0 & (t-t_0 \geq 0) \\ 0 & (t-t_0 < 0), \end{cases} \\ r(-t-t_0) &= \begin{cases} -t-t_0 & (-t-t_0 \geq 0) \\ 0 & (-t-t_0 < 0), \end{cases} \\ r(-t+t_0) &= \begin{cases} -t+t_0 & (-t+t_0 \geq 0) \\ 0 & (-t+t_0 < 0), \end{cases} \end{aligned}$$

and draw different ramp functions as shown in Figure 1.6.

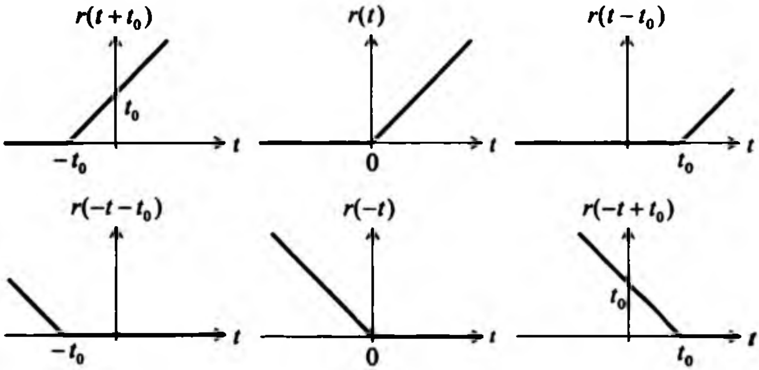


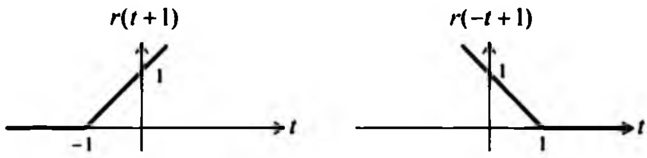
Figure 1.6: Unit Ramp functions

Example 1.3 Sketch time signal $x(t)$ that is given as

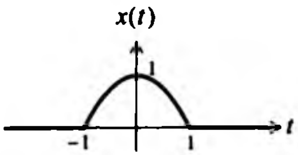
$$x(t) = r(t+1)\, r(-t+1).$$

Solution

We first sketch $r(t+1)$ and $r(-t+1)$ as follows:



It is obvious that for $t < -1$, $r(t+1)$ is zero, and $x(t)$ is thus zero. Likewise, for $t > 1$, $r(-t+1)$ is zero, and $x(t)$ is zero too. For $-1 < t < 1$, on the other hand, $x(t) = (t+1)(-t+1) = -t^2 + 1$. We therefore sketch $x(t)$ as follows:



1.4.2 Unit Step Function

The *unit step function* $u(t)$, also known as *Heaviside function*, is defined as

$$u(t) = \begin{cases} 1 & (t > 0), \\ 0 & (t < 0). \end{cases} \tag{1.2}$$

Note that $u(t)$ is discontinuous and undefined at $t = 0$. Utilizing the above definition, we can write the following relations:

$$u(t + t_0) = \begin{cases} 1 & (t + t_0 > 0) \\ 0 & (t + t_0 < 0), \end{cases}$$

$$u(t - t_0) = \begin{cases} 1 & (t - t_0 > 0) \\ 0 & (t - t_0 < 0), \end{cases}$$

$$u(-t-t_0) = \begin{cases} 1 & (-t-t_0 > 0) \\ 0 & (-t-t_0 < 0), \end{cases}$$

$$u(-t+t_0) = \begin{cases} 1 & (-t+t_0 > 0) \\ 0 & (-t+t_0 < 0), \end{cases}$$

and draw a series of step functions as shown in Figure 1.7. Step functions are frequently used to represent an abrupt change, like the changes that occur in circuits of control systems and digital computers.

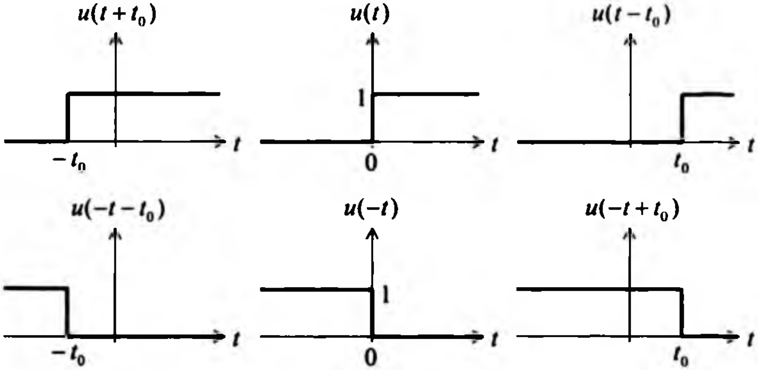
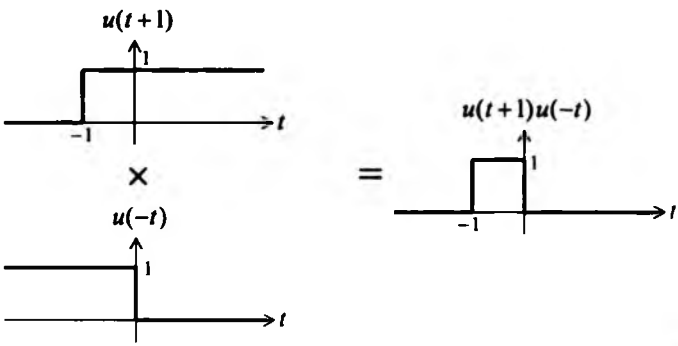


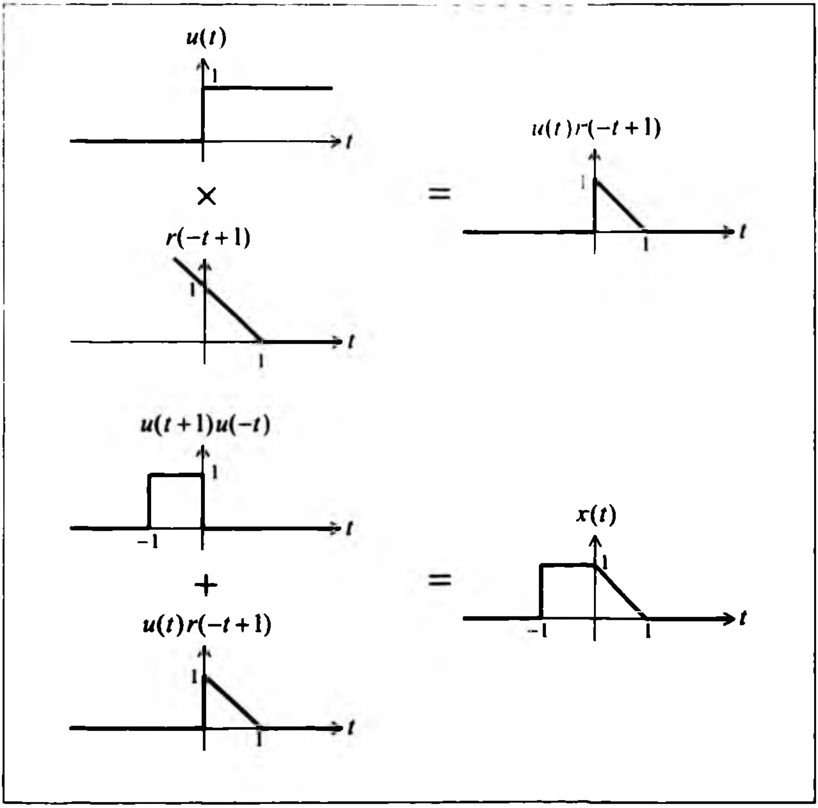
Figure 1.7: Unit step functions

Example 1.4 Sketch the time signal $x(t)$ that is given as

$$x(t) = u(t+1)\,u(-t) + u(t)\,r(-t+1).$$

Solution





1.4.3 Unit Impulse Function

The *unit impulse function* $\delta(t)$, also known as the *delta function*, is defined as

$$\delta(t) = \begin{cases} \infty & (t = 0), \\ 0 & (t \neq 0). \end{cases} \tag{1.3}$$

As the definition illustrates, the unit impulse function is strongly non-intuitive. To grasp the physical significance of the function more conveniently, consider rectangular functions shown in Figure 1.8 (a). Four different functions are shown, but the area above the time axis remains the same. We can now imagine a function that has infinite height, zero width, and an area that is identical to the area of the rectangular functions shown in Figure 1.8 (a). That is the unit impulse function. It is, however,

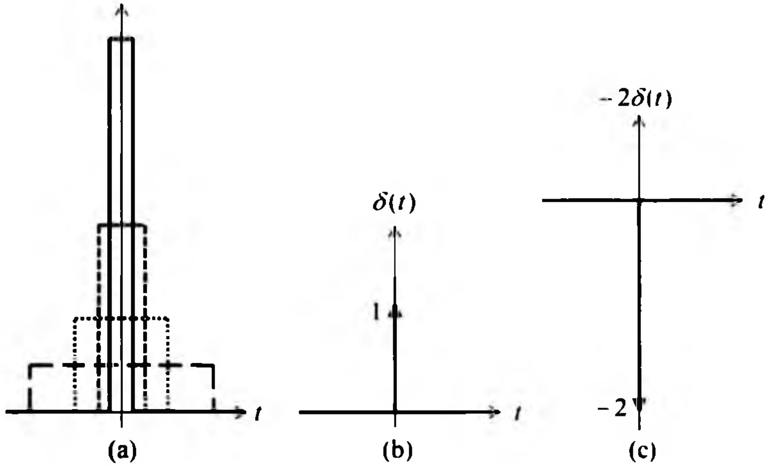


Figure 1.8: Concept of the unit impulse function

impossible to display the infinite height. We thus sketch the unit impulse function via the arrow symbol with the height of the arrow representing the area or integration value of impulse functions (Figures 1.8 (b) and (c)). Utilizing expression 1.3, we write the following relations:

$$\delta(t+t_0) = \begin{cases} \infty & (t+t_0=0) \\ 0 & (t+t_0 \neq 0), \end{cases}$$

$$\delta(t-t_0) = \begin{cases} \infty & (t-t_0=0) \\ 0 & (t-t_0 \neq 0), \end{cases}$$

$$\delta(-t-t_0) = \begin{cases} \infty & (-t-t_0=0) \\ 0 & (-t-t_0 \neq 0), \end{cases}$$

$$\delta(-t+t_0) = \begin{cases} \infty & (-t+t_0=0) \\ 0 & (-t+t_0 \neq 0). \end{cases}$$

With the above expressions, different impulse functions can readily be sketched as shown in Figure 1.9.

The unit impulse function $\delta(t)$ has several important properties. First

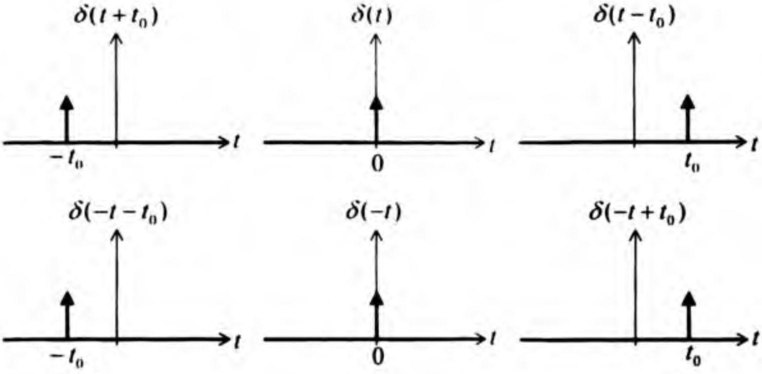


Figure 1.9: Unit impulse functions

of all, the integral of unit impulse function has the following properties:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1, \tag{1.4}$$

$$\int_{a}^{b} \delta(t - t_0)\, dt = \begin{cases} 1 & (a < t_0 < b), \\ 0 & (t_0 < a < b), \\ 0 & (a < b < t_0). \end{cases} \tag{1.5}$$

The impulse function can be also associated with other functions. Consider the following integral:

$$\int_{a}^{b} x(t)\, \delta(t - t_0)\, dt,$$

where $a < t_0 < b$. Since $\delta(t - t_0) = 0$ except at $t = t_0$, $x(t)\,\delta(t - t_0)$ must be also zero except at t_0. Thus

$$\int_{a}^{b} x(t)\, \delta(t - t_0)\, dt = \int_{a}^{b} x(t_0)\, \delta(t - t_0)\, dt = x(t_0) \int_{a}^{b} \delta(t - t_0)\, dt,$$

and we simplify the above expression as

$$\int_{a}^{b} x(t)\, \delta(t - t_0)\, dt = x(t_0). \tag{1.6}$$

Expression 1.6 shows that when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs. This is a useful property known as the *sampling* or *sifting property* of the impulse function. Note that the sifting property of the impulse function only makes sense upon integration, and do not be confused with the following expression:

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0) = \begin{cases} \infty & (t = t_0), \\ 0 & (t \neq t_0). \end{cases} \tag{1.7}$$

1.4.4 Relationship Between Singularity Functions

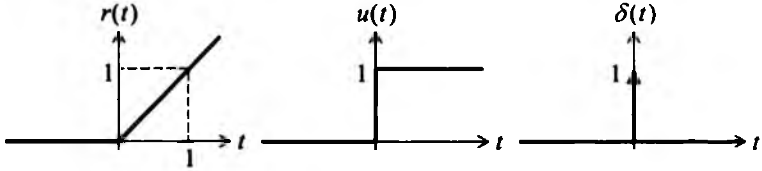


Figure 1.10: Basic continuous-time signals

The unit ramp function $r(t)$, unit step function $u(t)$, and unit impulse function $\delta(t)$ have the following differential / integral relations:

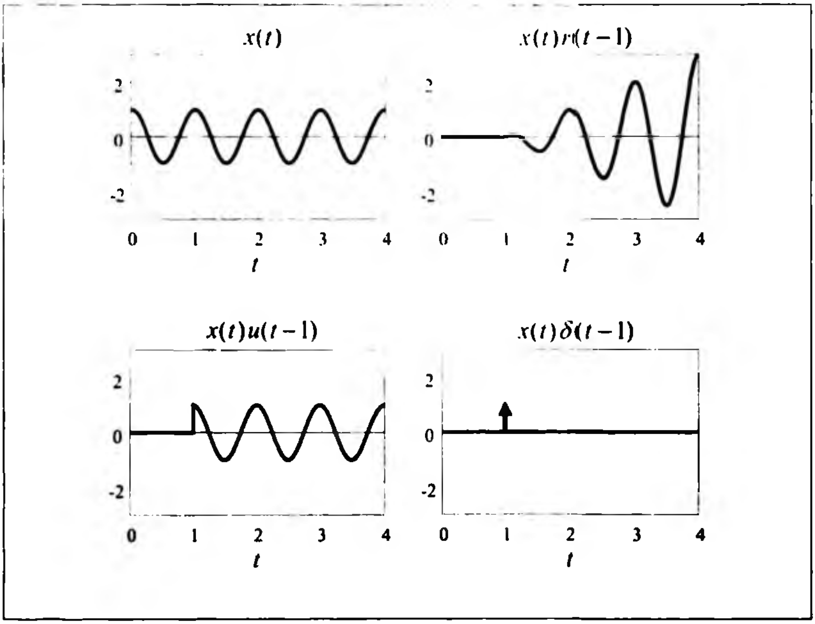
$$u(t) = \frac{dr(t)}{dt} \quad \text{and} \quad r(t) = \int_{-\infty}^{t} u(\tau)\,d\tau, \tag{1.8}$$

$$\delta(t) = \frac{du(t)}{dt} \quad \text{and} \quad u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau. \tag{1.9}$$

Interestingly, the unit ramp and unit step functions are also associated as $r(t) = t\,u(t)$.

Example 1.5 Assume $x(t) = \cos(2\pi t)$, and sketch $x(t)$, $x(t)\,r(t-1)$, $x(t)\,u(t-1)$, and $x(t)\,\delta(t-1)$, respectively.

Solution



1.5 BASIC DISCRETE-TIME SIGNALS

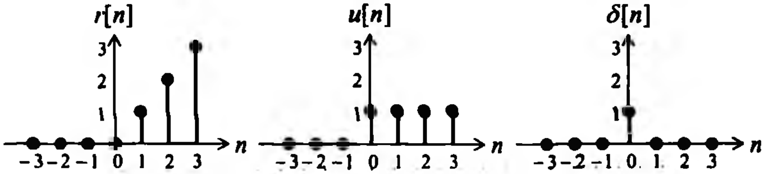


Figure 1.11: Basic discrete-time signals

We introduce several basic discrete-time signals. The *unit ramp sequence* $r[n]$ is defined as

$$r[n] = \begin{cases} n & (n \geq 0), \\ 0 & (n < 0), \end{cases} \tag{1.10}$$

the *unit step sequence* $u[n]$ as

$$u[n] = \begin{cases} 1 & (n \geq 0), \\ 0 & (n < 0), \end{cases} \tag{1.11}$$

and the *unit impulse sequence* $\delta[n]$ as

$$\delta[n] = \begin{cases} 1 & (n = 0), \\ 0 & (n \neq 0). \end{cases} \tag{1.12}$$

Note that unlike the unit step function $u(t)$ and unit impulse function $\delta(t)$, which are undefined at $t = 0$, the unit step sequence $u[n]$ and unit impulse sequence $\delta[n]$ are defined at $n = 0$. Note also that these three time sequences have the following difference / summation relations:

$$u[n] = r[n+1] - r[n] \quad \text{and} \quad r[n] = \sum_{k=-\infty}^{n-1} u[k], \tag{1.13}$$

$$\delta[n] = u[n] - u[n-1] \quad \text{and} \quad u[n] = \sum_{k=-\infty}^{n} \delta[k]. \tag{1.14}$$

Incidentally, the unit ramp and unit step sequences are also associated as $r[n] = n\,u[n]$.

The unit impulse sequence has several important properties that deserve our special attention. First of all, it is evident from the definition that changing the sign of index does not alter an impulse sequence:

$$\delta[n-k] = \delta[k-n]. \tag{1.15}$$

We next consider multiplying two impulse sequences as shown in Figure 1.12. It is clear that $\delta[n]\,\delta[n-1]$ is always zero. Similarly, we can

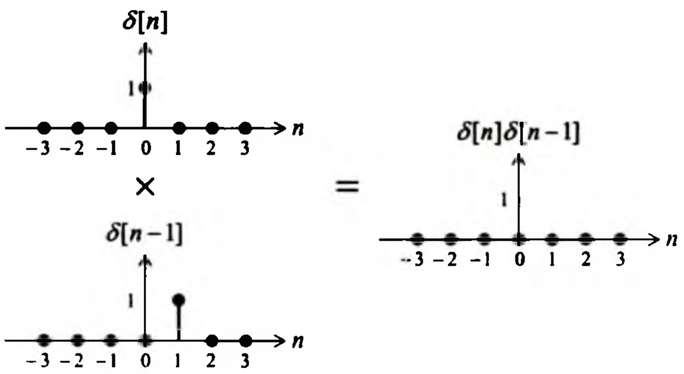


Figure 1.12: Multiplication of two impulse sequences

easily conclude that $\delta[n]\,\delta[n] = \delta[n]$ and $\delta[n-1]\,\delta[n-1] = \delta[n-1]$. We thus deduce a property of the impulse sequence as follows:

$$\delta[n-k_1]\,\delta[n-k_2] = \begin{cases} \delta[n-k] & (k_1 = k_2 = k), \\ 0 & (k_1 \neq k_2). \end{cases} \tag{1.16}$$

Another important property of the impulse sequence is the *sampling* or *sifting property*. Consider multiplying two sequences in Figure 1.13. It is evident that $x[n]\,\delta[n-1] = x[1]\,\delta[n-1]$, and we can generally write the following expression:

$$x[n]\,\delta[n-k] = x[k]\,\delta[n-k]. \tag{1.17}$$

The sifting property enables one to expand $x[n]$ as

$$\begin{aligned} x[n] = \cdots &+ x[-2]\,\delta[n+2] + x[-1]\,\delta[n+1] + x[0]\,\delta[n] \\ &+ x[1]\,\delta[n-1] \quad + x[2]\,\delta[n-2] \quad + \cdots, \end{aligned}$$

or, in short, as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]. \tag{1.18}$$

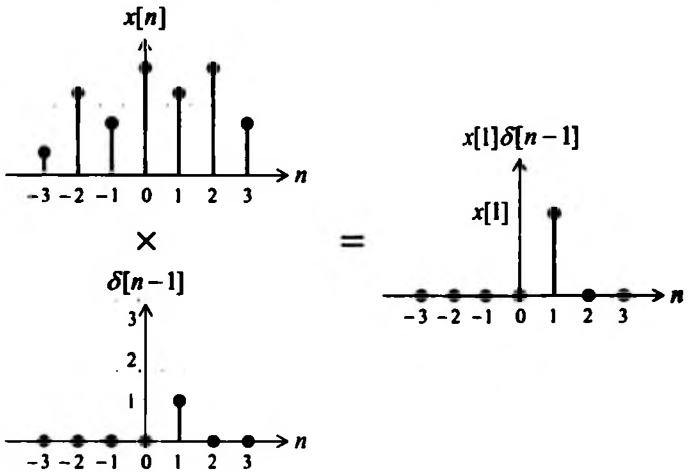


Figure 1.13: Sifting property of the impulse sequence

Substituting $r[n]$ or $u[n]$ into $x[n]$, we can also derive that

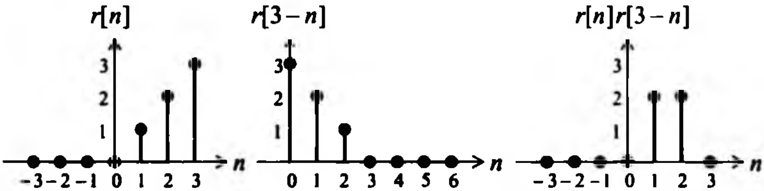
$$r[n] = \sum_{k=-\infty}^{\infty} r[k]\,\delta[n-k] = \sum_{k=1}^{\infty} k\,\delta[n-k], \tag{1.19}$$

and

$$u[n] = \sum_{k=-\infty}^{\infty} u[k]\,\delta[n-k] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{1.20}$$

Example 1.6 It can be shown in a graphical way that

$$r[n]\,r[3-n] = 2\delta[n-1] + 2\delta[n-2].$$



Perform an algebraic evaluation of $r[n]\,r[3-n]$ and derive the result shown above.

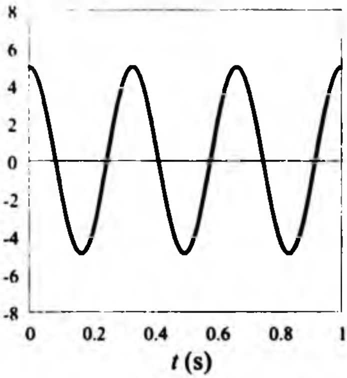
Solution

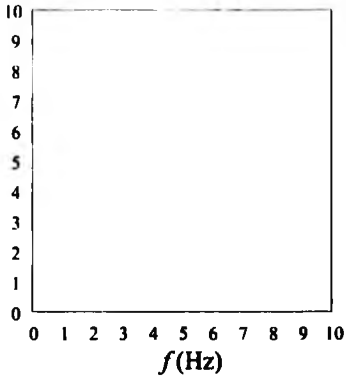
$$\begin{aligned} r[n]\,r[3-n] &= \left(\sum_{k=1}^{\infty} k\,\delta[n-k]\right)\left(\sum_{k=1}^{\infty} k\,\delta[3-n-k]\right) \\ &= \left(\sum_{k=1}^{\infty} k\,\delta[n-k]\right)\left(\sum_{k=1}^{\infty} k\,\delta[n+k-3]\right) \end{aligned}$$

$$\begin{aligned} &= (\delta[n-1] + 2\delta[n-2] + \cdots)\,(\delta[n-2] + 2\delta[n-1] + \cdots) \\ &= 2\delta[n-1]\,\delta[n-1] + 2\delta[n-2]\,\delta[n-2] \\ &= 2\delta[n-1] + 2\delta[n-2]. \end{aligned}$$

PROBLEMS

Problem 1.1 Consider the time signal shown below. Sketch the amplitude spectrum of the time signal.

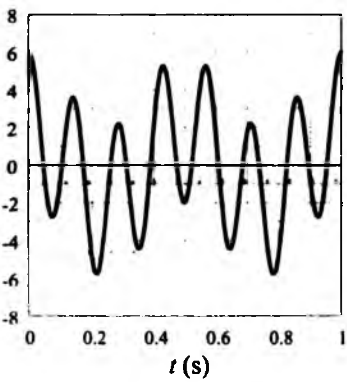


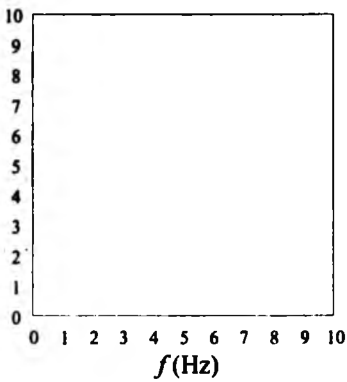


Problem 1.2 Consider the time signal shown below. The signal has two frequency components:

$$x(t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t).$$

Sketch the amplitude spectrum of the time signal.





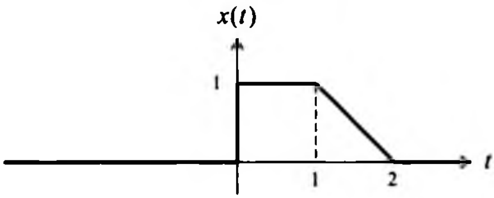
Problem 1.3 Sketch the following function:

$$x(t) = r(t+2)\, r(-t+2).$$

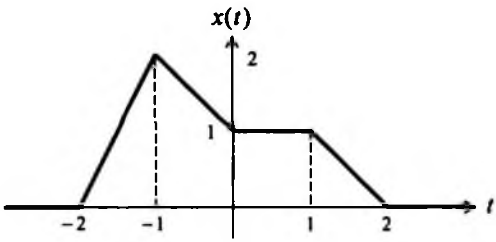
Problem 1.4 Sketch the following function:

$$x(t) = 2r(t)\, u(-t+1) + 2r(-t+2)\, u(t-1).$$

Problem 1.5 Give a mathematical expression of the function $x(t)$ as a combination of basic continuous-time signals.



Problem 1.6 Give a mathematical expression of the function $x(t)$ as a combination of basic continuous-time signals.



Problem 1.7 Which of the following expressions is correct? Choose one.

a. $\delta(t-t_0) = \dfrac{d}{dt} r(t-t_0)$

b. $\delta(t) = -\delta(-t)$

c. $\delta(t-t_0) = \delta(-t-t_0)$

d. $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

Problem 1.8 Which of the following expressions is wrong? Choose one.

a. $u(t - t_0) = \frac{d}{dt}\delta(t - t_0)$

b. $\delta(t) = \delta(-t)$

c. $\delta(t - t_0) = \delta(-t + t_0)$

d. $x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$

Problem 1.9 Express the following sequence as a linear combination of unit impulse sequences (Don't use graphic approaches! Just do algebraic calculation!):

$$x[n] = r[n+2]\,r[-n+2].$$

Problem 1.10 Express the following sequence as a linear combination of unit impulse sequences (Don't use graphic approaches! Just do algebraic calculation!):

$$x[n] = u[n+1]\,r[-n+3].$$

Chapter 2

CLASSIFICATIONS OF SIGNALS

There are many ways of classifying signals: continuous-time or discrete-time, analog or digital, even or odd, periodic or nonperiodic, energy or power, random or non-random, real or complex, etc. The topic about continuous-time and discrete-time signals has been covered in Chapter 1. And, in Chapter 2, we discuss characteristics of analog and digital signals, even and odd signals, periodic and nonperiodic signals, and energy and power signals.

2.1 ANALOG AND DIGITAL SIGNALS

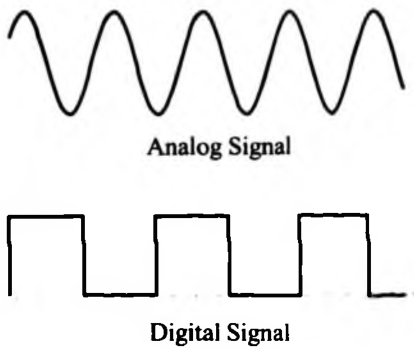


Figure 2.1: An analog signal and a digital signal

The term "analog signal" is frequently used to mean any continuous-time signal. There is, however, a distinction between the two. An *analog signal* is a continuous-time signal for which the time-varying feature of the signal is a representation of a physical phenomenon. All analog signals are continuous-time signals, but all continuous-time signals are not analog signals. Similarly, there exists a distinction between "digital signal" and discrete-time signal. A *digital signal* is a discrete-time signal that can have only a finite number of values (usually binary).

The world we are living in is fundamentally an analog world, and most signals are thus analog. Consider, for example, taking a signal from a microphone and recording the signal into a magnetic tape. The signal from the microphone is definitely an analog signal, and the signal on the tape is also analog. In an analog system, however, it is generally difficult to remove noise and to avoid signal distortions during the data transmission. And, as a result of that, analog signals are inappropriate for high quality data transmission. Digital signals, on the other hand, use binary data strings (0 and 1) and allow one to achieve high quality data transmission. Most of the time, we obtain digital signals from analog signals via an *analog-to-digital converter* (ADC).

2.2 EVEN AND ODD SIGNALS

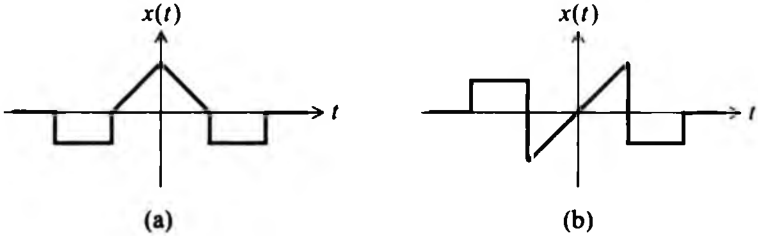


Figure 2.2: An even function (a) and odd function (b)

By definition, an *even signal* is described as

$$x(t) = x(-t) \qquad \text{(even function)} \tag{2.1}$$

or

$$x[n] = x[-n] \qquad \text{(even sequence),} \tag{2.2}$$

and an *odd signal* as

$$x(t) = -x(-t) \qquad \text{(odd function)} \tag{2.3}$$

or

$$x[n] = -x[-n] \qquad \text{(odd sequence).} \tag{2.4}$$

For example, $x(t) = \cos(2\pi t)$ is an even function, while $x[n] = \sin(n\pi/12)$ is an odd sequence. Figure 2.2 illustrates examples of even and odd functions.

An interesting fact about even and odd symmetry is that any signal can be decomposed into two parts: one having even symmetry and the other having odd symmetry. In other words, any time function $x(t)$ can be expressed as

$$x(t) = x_e(t) + x_o(t), \tag{2.5}$$

where the even part $x_e(t)$ and odd part $x_o(t)$ are

$$x_e(t) = \frac{x(t) + x(-t)}{2}, \tag{2.6}$$

and

$$x_o(t) = \frac{x(t) - x(-t)}{2}. \tag{2.7}$$

Likewise, one may express any time sequence $x[n]$ as

$$x[n] = x_e[n] + x_o[n], \tag{2.8}$$

with

$$x_e[n] = \frac{x[n] + x[-n]}{2}, \tag{2.9}$$

and

$$x_o[n] = \frac{x[n] - x[-n]}{2}. \tag{2.10}$$

Figure 2.3 demonstrates an example of decomposing time sequence into the sum of even and odd sequences. Note also the following properties of even and odd signals:

1. Adding / subtracting two even signals yields an even signal.
2. Adding / subtracting two odd signals yields an odd signal.
3. Adding / subtracting an even signal and an odd signal yields a signal that is neither even nor odd.
4. Multiplying two even signals gives an even signal.
5. Multiplying two odd signals gives an even signal.
6. Multiplying an even signal and an odd signal gives an odd signal.

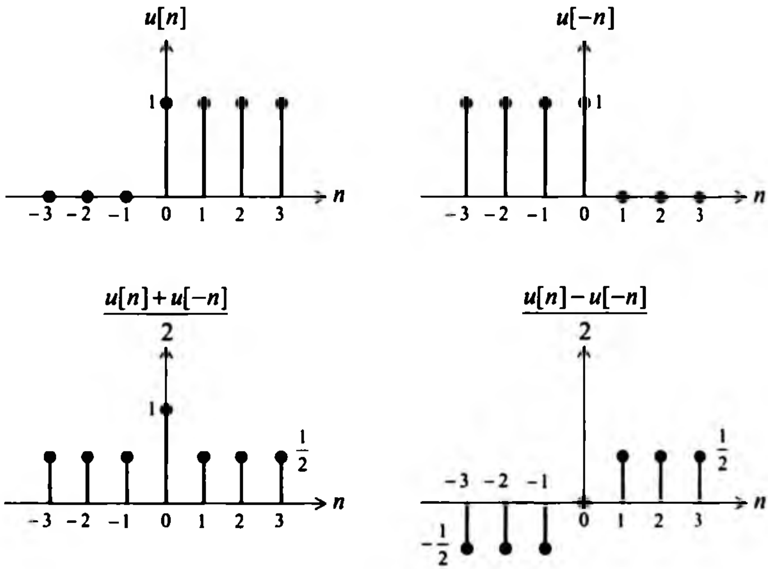


Figure 2.3: The unit step sequence decomposed into the sum of the even and odd sequences

Example 2.1 Derive the even and odd parts of the following time function:

$$x(t) = 10\sin(t) - 5\cos(2t) + 2\sin(5t)\cos(5t).$$

Solution

$$\begin{aligned} x(-t) &= 10\sin(-t) - 5\cos(-2t) + 2\sin(-5t)\cos(-5t) \\ &= -10\sin(t) - 5\cos(2t) - 2\sin(5t)\cos(5t). \end{aligned}$$

$$\begin{aligned} x(t) + x(-t) &= -10\cos(2t), \\ x(t) - x(-t) &= 20\sin(t) + 4\sin(5t)\cos(5t). \end{aligned}$$

$$\begin{aligned} x_e &= -5\cos(2t), \\ x_o &= 10\sin(t) + 2\sin(5t)\cos(5t). \end{aligned}$$

2.3 PERIODIC AND NONPERIODIC SIGNALS

A signal is a *periodic signal* if it completes a pattern within a measurable time frame, called the period and repeats that pattern over identical subsequent periods. A *nonperiodic signal*, on the other hand, is the one that does not exhibit a repeating pattern. Basic concept of the periodicity is identical for continuous-time and discrete-time signals, but detailed properties differ significantly. We therefore study periodic functions and periodic sequences separately and later discuss those differences in detail.

2.3.1 Periodic and Nonperiodic Functions

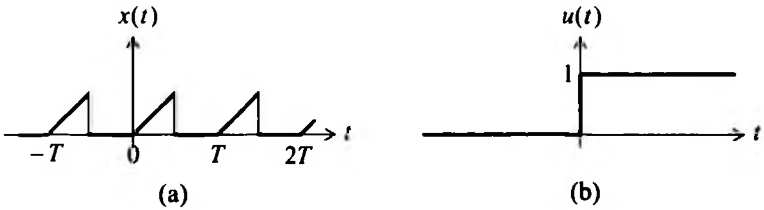


Figure 2.4: A periodic function (a) and nonperiodic function (b)

A continuous-time signal is called periodic if it satisfies

$$x(t) = x(t + kT), \tag{2.11}$$

where k is an integer, and T denotes the period of the time function. Figure 2.4 exemplifies periodic and nonperiodic functions. Note that for a periodic function with period T, one may safely argue that the function is also periodic with a period $2T$, $3T$, or kT, where k is a natural number. Among these different values of period, the smallest one has the greatest significance. To avoid confusion, we call the smallest possible period the *fundamental period* of the signal and denote as T_0.

Scientists and engineers encounter a variety of different periodic functions while solving problems at their hand. Among those periodic functions, sinusoidal functions deserve our great attention, because, as detailed in Chapter 7, sinusoidal functions are the basic building block from which one can construct any periodic functions. A *sinusoidal function* is expressed as

$$x(t) = A\cos(\omega t + \varphi), \tag{2.12}$$

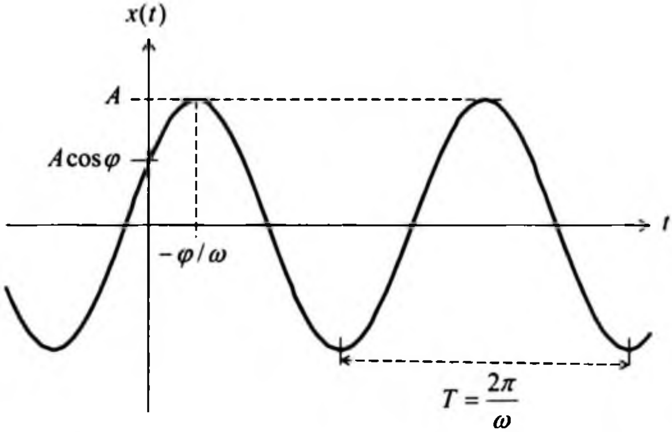


Figure 2.5: Geometry of a sinusoidal function

with

$$\omega = 2\pi f = \frac{2\pi}{T}, \tag{2.13}$$

where A, T, f, ω, and φ represent the amplitude, period, frequency, angular frequency, and phase of the function, respectively.

Linear combinations of two sinusoidal functions are, however, not always periodic. Consider a time function of the following form:

$$x(t) = A_1 \cos(2\pi t/T_1 + \varphi_1) + A_2 \cos(2\pi t/T_2 + \varphi_2).$$

The signal is a linear combination of two sinusoidal functions whose periods are T_1 and T_2, respectively. The fundamental period T_0 of $x(t)$ is then expressed as

$$T_0 = k_1 T_1 = k_2 T_2, \tag{2.14}$$

with an assumption that integers k_1 and k_2 do exist and one chooses the smallest possible values of them. To better understand the existence of k_1 and k_2, consider the following relation:

$$\frac{k_1}{k_2} = \frac{T_2}{T_1}. \tag{2.15}$$

The above expression shows that $x(t)$ is periodic (i.e., integers k_1 and k_2 exist) only when the number T_2/T_1 is a rational number. Incidentally, the *fundamental frequency* is the angular frequency that corresponds to the fundamental period, and we denote the fundamental frequency as Ω such that

$$\Omega = \frac{2\pi}{T_0}. \tag{2.16}$$

Example 2.2 Determine the periodicity of

$$x(t) = 2\cos(3t/5 + 1) - \sin(9t/4 - 1),$$

and, if periodic, find the fundamental period T_0.

Solution

Denote

$$x(t) = x_1(t) + x_2(t),$$

with

$$x_1(t) = 2\cos(3t/5 + 1),$$

and

$$x_2(t) = -\sin(9t/4 - 1).$$

Angular frequencies ω_1 and ω_2 are $3/5$ and $9/4$, respectively. Therefore

$$T_1 = \frac{2\pi}{\omega_1} = \frac{10\pi}{3}, \quad T_2 = \frac{2\pi}{\omega_2} = \frac{8\pi}{9},$$

and

$$\frac{T_2}{T_1} = \frac{24}{90} = \frac{4}{15} = \frac{k_1}{k_2}.$$

Therefore, $x(t)$ is periodic and its fundamental period T_0 is given as

$$T_0 = 4T_1 = 15T_2 = \frac{40\pi}{3}.$$

Example 2.3 Determine the periodicity of

$$x(t) = \cos(3t + 1) - \sin(\pi t - 1),$$

and, if periodic, find the fundamental period T_0.

Solution

Denote

$$x(t) = x_1(t) + x_2(t),$$

with

$$x_1(t) = \cos(3t + 1),$$

and

$$x_2(t) = -\sin(\pi t - 1).$$

Angular frequencies ω_1 and ω_2 are 3 and π, respectively. Therefore

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{3}, \quad T_2 = \frac{2\pi}{\omega_2} = 2,$$

and

$$\frac{T_2}{T_1} = \frac{6}{2\pi} = \frac{3}{\pi} \neq \frac{k_1}{k_2}.$$

The number $3/\pi$ is an irrational number, and $x(t)$ is not periodic.

2.3.2 Periodic and Nonperiodic Sequences

A discrete-time signal is called a periodic sequence if it satisfies

$$x[n] = x[n + kN], \tag{2.17}$$

where k is an integer, and another integer N is the period of the time sequence. Figure 2.6 exemplifies periodic and nonperiodic sequences.

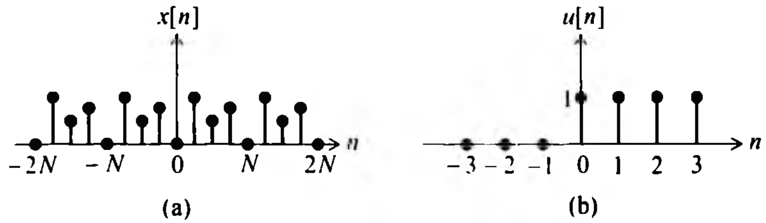


Figure 2.6: A periodic sequence (a) and nonperiodic sequence (b)

Similarly to expression 2.12, a *sinusoidal sequence* is expressed as

$$x[n] = A\cos(\omega n + \varphi). \tag{2.18}$$

However, unlike sinusoidal functions, sinusoidal sequences in expression 2.18 are not always periodic. The reason is because the cosine function repeats itself at every time increase / decrease of 2π (i.e., $\cos(t) = \cos(t \pm 2\pi)$). On the other hand, with the integer variable n, one cannot guarantee for sure that ωn in expression 2.18 can be a multiple of 2π. For example, with a rational number ω, ωn can never be a multiple of 2π. In other words, only ω being expressed as $\rho\pi$ with a rational number ρ, $x[n]$ is periodic, and the *fundamental period* N_0 of expression 2.18 is expressed as

$$N_0 = \frac{2\pi k}{\omega} = \frac{k}{f}, \tag{2.19}$$

with the smallest possible natural number k.

A linear combination of two periodic sinusoidal sequences is always periodic. And its fundamental period is given as the *least common multiple* (LCM) of each of the two periods N_1 and N_2:

$$N_0 = \mathrm{LCM}(N_1, N_2). \tag{2.20}$$

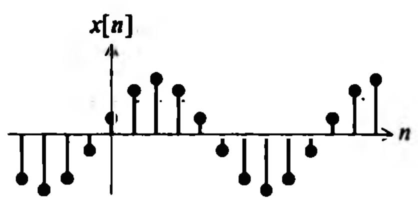


Figure 2.7: A "periodic" sinusoidal sequence: $x[n] = A\cos(\pi(n-2)/5)$

Example 2.4 Determine the periodicity of

$$x[n] = 2\cos(3\pi(n+1)/5),$$

and, if periodic, find the fundamental period N_0.

Solution

$$\omega = \frac{3\pi}{5}, \quad f = \frac{\omega}{2\pi} = \frac{3}{10}, \quad \text{and} \quad N_0 = \frac{k}{f} = \frac{10k}{3}.$$

The smallest natural number k that yields an integer N_0 is 3. Thus, $x[n]$ is periodic with the fundamental frequency $N_0 = 10$.

Example 2.5 Determine the periodicity of

$$x[n] = 3\cos(5(n-1)/2),$$

and, if periodic, find the fundamental period N_0.

Solution

$$\omega = \frac{5}{2}, \quad f = \frac{\omega}{2\pi} = \frac{5}{4\pi}, \quad \text{and} \quad N_0 = \frac{k}{f} = \frac{4\pi k}{5}.$$

π is an irrational number. Therefore there exists no natural number k that yields an integer N_0. In other words, $x[n]$ is not periodic.

Example 2.6 Determine the periodicity of

$$x[n] = \cos(\pi n/3) + \sin(5\pi n/7),$$

and, if periodic, find the fundamental period N_0.

Solution

Denote

$$x[n] = x_1[n] + x_2[n],$$

with

$$x_1[n] = \cos(\pi n/3),$$

and

$$x_2[n] = \sin(5\pi n/7) = \cos(5\pi n/7 - \pi/2).$$

With the first term $x_1[n]$,

$$\omega_1 = \frac{\pi}{3}, \quad f_1 = \frac{\omega_1}{2\pi} = \frac{1}{6}, \quad \text{and} \quad N_1 = \frac{k_1}{f_1} = 6k_1.$$

The smallest natural number k_1 that yields an integer N_1 is 1, and $N_1 = 6$. With the second term $x_2[n]$,

$$\omega_2 = \frac{5\pi}{7}, \quad f_2 = \frac{\omega_2}{2\pi} = \frac{5}{14}, \quad \text{and} \quad N_2 = \frac{k_2}{f_2} = \frac{14k_2}{5}.$$

The smallest natural number k_2 that yields an integer N_2 is 5, and $N_2 = 14$. Finally, the fundamental period of $x[n]$ is given as

$$N_0 = \text{LCM}(N_1, N_2) = \text{LCM}(6, 14) = 42.$$

2.3.3 Sinusoidal Functions and Sequences

It has been demonstrated in the previous subsections that sinusoidal sequence may be a lot different from sinusoidal function in its character. We thus consider the following simple expressions:

$$x(t) = \cos(\omega t) \quad \text{and} \quad x[n] = \cos(\omega n),$$

as prototypes of sinusoidal functions and sequences, and focus on discussing differences between them.

Periodicity in time domain

The time function $x(t)$ is always periodic as a function of time t, and the fundamental period of the signal is

$$T_0 = \frac{2\pi}{\omega}.$$

The time sequence $x[n]$, on the other hand, may not be periodic as a function of time n. It is periodic only when $\omega = \rho\pi$ (with a rational number ρ), and the fundamental period of the signal is given as

$$N_0 = \frac{2\pi k}{\omega},$$

with the smallest possible natural number k.

Periodicity in frequency domain

Discussing periodicity in frequency domain is to argue if we observe any repeating pattern while we change ω to $\omega + \omega_0$. Sinusoidal functions are never periodic as functions of frequency:

$$\cos((\omega + \omega_0)\,t) \neq \cos(\omega t).$$

In other words, changing frequency always yields different signal. Sinusoidal sequences are, on the other hand, always periodic in frequency domain with $\omega_0 = 2\pi$:

$$\cos((\omega + 2\pi)\,n) = \cos(\omega n + 2\pi n) = \cos(\omega n).$$

It may sound puzzling at first, but this consequence arises from the basic nature of discrete signals that n must be an integer. This interesting property of sinusoidal sequences will be revisited in Chapter 11.

Frequency and oscillation

Related to the periodicity in frequency domain, we can easily argue that for sinusoidal functions, increasing frequency means faster oscillation. It is, however, not the case for sinusoidal sequences. Figure 2.8 demonstrates that $x[n]$ exhibits no oscillation with $\omega = 0$. As ω increases to $\pi/12$, $\pi/6$, and finally upto π, $x[n]$ shows faster and faster oscillation. Exceeding π, however, does not accompany stronger oscillation. Instead, the oscillation

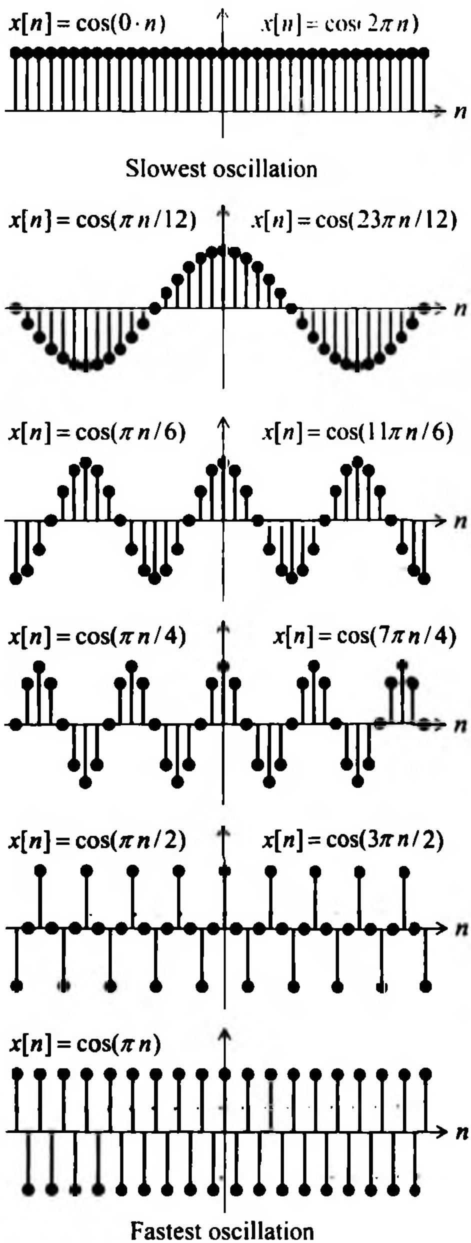


Figure 2.8: Sinusoidal sequences $x[n] = \cos(\omega n)$ with different values of angular frequency ω

gets slower as ω increases from π to 2π. In short, increasing frequency does not always accompany faster oscillation. And, in connection to the repeating pattern in frequency domain, we conclude that sinusoidal sequences exhibit the strongest oscillation when $\omega = (2k+1)\pi$ and the slowest oscillation (in other word, no oscillation) when $\omega = 2k\pi$.

2.4 ENERGY AND POWER SIGNALS

For a continuous-time signal $x(t)$, *total energy* is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt, \tag{2.21}$$

and *average power* as

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt. \tag{2.22}$$

For a discrete-time signal $x[n]$, total energy is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2, \tag{2.23}$$

and average power as

$$P = \lim_{k\to\infty} \frac{1}{2k+1} \sum_{n=-k}^{k} |x[n]|^2. \tag{2.24}$$

Based on the definitions of E and P, we further define the following:

- A signal $x(t)$ or $x[n]$ is an *energy signal* if and only if $0 < E < \infty$ and consequently $P = 0$.
- A signal $x(t)$ or $x[n]$ is a *power signal* if and only if $E = \infty$ and $0 < P < \infty$.

If a signal is a power signal, then it cannot be an energy signal or vice versa; energy and power signals are mutually exclusive. A signal may be neither an energy nor a power signal, and almost all periodic functions of practical interest are power signals.

Example 2.7 Determine whether the following signal is an energy signal, power signal, or neither of them.

$$x[n] = \begin{cases} (1/2)^n & (n \geq 0), \\ 0 & (\text{otherwise}). \end{cases}$$

Solution

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} (1/2)^{2n} = \sum_{n=0}^{\infty} (1/4)^n = \frac{1}{1 - 1/4} = \frac{4}{3}.$$

E is finite, and $x[n]$ is thus an energy signal.

Example 2.8 Determine whether the following signal is an energy signal, power signal, or neither of them.

$$x(t) = u(t).$$

Solution

$$E = \int_{-\infty}^{\infty} |u(t)|^2 \, dt = \int_0^{\infty} 1 \, dt = \infty.$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |u(t)|^2 \, dt = \lim_{T \to \infty} \frac{1}{2T} \int_0^T 1 \, dt$$

$$= \lim_{T \to \infty} \frac{T}{2T} = \frac{1}{2}.$$

$x(t)$ is thus a power signal.

Example 2.9 Determine whether the following signal is an energy signal, power signal, or neither of them.

$$x(t) = \sin(t).$$

Solution

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} \sin^2(t)\,dt = \infty.$$

$$\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sin^2(t)\,dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} \sin^2(t)\,dt = \lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} 1 - \cos(2t)\,dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\left[T - \frac{1}{2}\sin(2T)\right] \\
&= \lim_{T\to\infty} \frac{T}{2T} - \lim_{T\to\infty} \frac{\sin(2T)}{4T} = \frac{1}{2}.
\end{aligned}$$

$x(t)$ is thus a power signal.

Example 2.10 Determine whether the following signal is an energy signal, power signal, or neither of them.

$$x[n] = r[n].$$

Hint: Refer to expression B.48.

Solution

$$E = \sum_{n=-\infty}^{\infty} |r[n]|^2 = \sum_{n=1}^{\infty} n^2 = \infty.$$

$$\begin{aligned}
P &= \lim_{k\to\infty} \frac{1}{2k+1}\sum_{n=-k}^{k} |r[n]|^2 = \lim_{k\to\infty} \frac{1}{2k+1}\sum_{n=1}^{k} n^2 \\
&= \lim_{k\to\infty} \frac{1}{2k+1}\frac{k(k+1)(2k+1)}{6} = \lim_{k\to\infty} \frac{k(k+1)}{6} = \infty.
\end{aligned}$$

$x[n]$ is thus neither an energy signal nor a power signal.

PROBLEMS

Problem 2.1 Determine the period T of the following signal:

$$x(t) = \sin(\pi t/2),$$

and sketch the signal for $0 < t < 2T$.

Problem 2.2 Determine the period T of the following signal:

$$x(t) = \cos(\pi t),$$

and sketch the signal for $0 < t < 2T$.

Problem 2.3 Find the fundamental period T_0 of the following signal:

$$x(t) = 3\cos^2(2\pi t/5) - 2\sin^2(\pi t/3).$$

Problem 2.4 Find the fundamental period T_0 of the following signal:

$$x(t) = 2\sin(3\pi t/4) + \sin^2(\pi t/5).$$

Problem 2.5 Find the fundamental period N_0 of the following signal:

$$x[n] = 2\cos(\pi(3n - 1)/5) + \cos(\pi(5n + 1)/3).$$

Problem 2.6 Find the fundamental period N_0 of the following signal:

$$x[n] = 2\cos(\pi(5n - 1)/7) - 3\cos(\pi(3n + 1)/5).$$

Problem 2.7 Consider the following signals: $x(t) = \cos(\omega t)$ and $x[n] = \cos(\omega n)$. Which of the following explanations is correct? Choose one.

a. $x(t)$ may not be periodic as a function of time.

b. Fundamental period N_0 of $x[n]$ is $2\pi/\omega$.

c. The highest oscillation of $x[n]$ occurs when $\omega = (2k+1)\pi$ with an integer k.

d. $x(t)$ may be periodic in frequency domain.

Problem 2.8 Consider the following signals: $x(t) = \cos(\omega t)$ and $x[n] = \cos(\omega n)$. Which of the following explanations is correct? Choose one.

a. $x[n]$ is always periodic as a function of time.

b. Fundamental period N_0 of $x[n]$ is $2\pi/\omega$.

c. The highest oscillation of $x(t)$ occurs when $\omega = (2k+1)\pi$ with an integer k.

d. $x(t)$ is never periodic in frequency domain.

Problem 2.9 The total energy E and average power P of a time function $x(t)$ are given as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \text{and} \quad P = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt,$$

respectively. Determine whether the following signal is an energy signal, power signal, or neither of them:

$$x(t) = e^{-t/2} u(t).$$

Problem 2.10 The total energy E and average power P of a time sequence $x[n]$ are given as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad \text{and} \quad P = \lim_{k \to \infty} \frac{1}{2k+1} \sum_{n=-k}^{k} |x[n]|^2,$$

respectively. Determine whether the following signal is an energy signal, power signal, or neither of them:

$$x[n] = \cos(3\pi n).$$

Chapter 3

OPERATIONS ON SIGNALS

As argued in Chapter 1, one interacts with a system via signals and identifies the nature of the system by observing how the system operates on signals. In Chapter 3, we summarize various operations a system may act on signals and their mathematical expressions. We also discuss how continuous-time and discrete-time signals may react in a significantly different way to an operation that is apparently similar for the two cases. We consider two representative types of operations: amplitude and time operations.

3.1 AMPLITUDE OPERATIONS

A system may influence the amplitude of an input signal $x(t)$, and we can generally express the operation of the system as

$$y(t) = Ax(t) + B, \tag{3.1}$$

where A and B are constants. Depending on values of A and B, we consider three possible cases: amplitude scaling, amplitude reversal, and amplitude shifting.

Amplitude scaling is expressed as $y(t) = Ax(t)$. Signals are being amplified with $|A| > 1$ and attenuated $|A| < 1$. Figure 3.1 is an example of amplitude scaling with $A = 2$. *Amplitude reversal* is expressed as $y(t) = -x(t)$, and signals are reflected about the horizontal axis. Figure 3.2 is an example of amplitude reversal. Note that for sinusoidal functions,

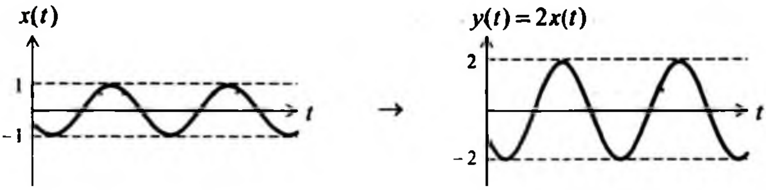


Figure 3.1: An example of amplitude scaling

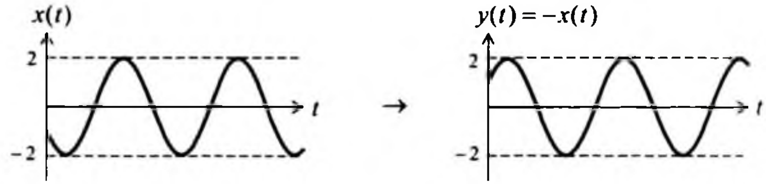


Figure 3.2: An example of amplitude reversal

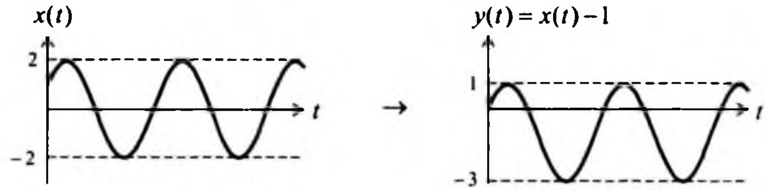


Figure 3.3: An example of amplitude shifting

amplitude reversal is identical to 180° phase shift:

$$\cos(\omega t + \varphi \pm \pi) = -\cos(\omega t + \varphi).$$

Amplitude shifting is expressed as $y(t) = x(t) + B$. Figure 3.3 is an example of amplitude shifting with $B = -1$.

All of the above amplitude operations can work simultaneously. Figure 3.4 shows that three operations presented in Figures 3.1 - 3.3 may occur at the same time ($A = -2$ and $B = -1$). While processing expression 3.1, one should must perform amplitude shifting later than amplitude scaling or reversal. Performing amplitude shifting prior to amplitude scaling / reversal causes a different result:

$$-2(x(t) - 1) = -2x(t) + 2 \neq -2x(t) - 1.$$

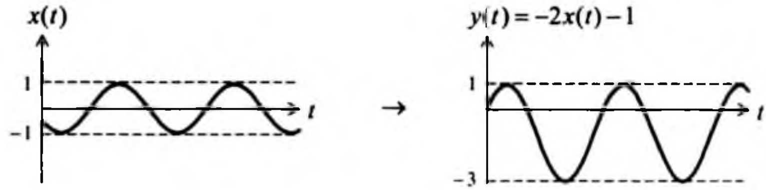


Figure 3.4: An example of combined operations on amplitude

We have so far considered amplitude operations on continuous-time signals. Previous discussions about amplitude operations are, however, not limited to continuous-time signals. Considering discrete-time signals, one can express amplitude operations as

$$y[n] = Ax[n] + B, \tag{3.2}$$

and utilize aforementioned conclusions without problem.

3.2 TIME OPERATIONS

A system may influence the way signals vary with time, and we can generally express the time operation of the system as

$$y(t) = x(at + b) \quad \text{(continuous-time)}, \tag{3.3}$$

$$y[n] = x[an + b] \quad \text{(discrete-time)}, \tag{3.4}$$

where a and b are constants. Depending on values of a and b, we may consider three cases: time shifting, time reversal, and time scaling.

3.2.1 Time Shifting

Time shifting is expressed as

$$y(t) = x(t + b) \quad \text{(continuous-time)}, \tag{3.5}$$

$$y[n] = x[n + b] \quad \text{(discrete-time)}, \tag{3.6}$$

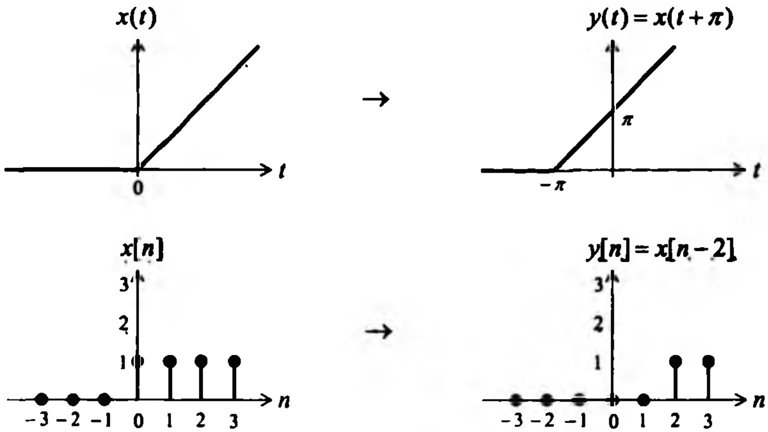


Figure 3.5: Examples of time shifting operations

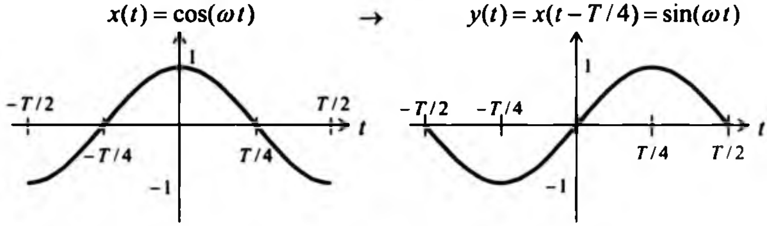


Figure 3.6: Time shifting of a sinusoidal function

with b being a real number for continuous-time and an integer for discrete-time signals. A positive value of b means signal is being advanced (or being shifted to the left), and a negative value of b means signals is being delayed (or being shifted to the right).

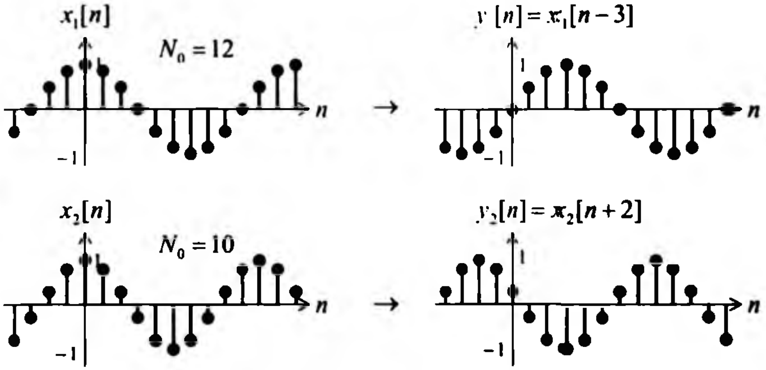
It is noteworthy that sinusoidal functions are always periodic and, therefore, a sinusoidal function may time shift to the original input function ($\cos(\omega t + 2\pi) = \cos(\omega t)$). Moreover, a cosine function may time shift to a sine function, and vice versa, because a sine function is the $T/4$ time delay (90° phase shift) of the cosine function whose frequency is identical to the sine function:

$$\begin{aligned}\cos\left(\omega\left(t-\frac{T}{4}\right)\right) &= \cos\left(\frac{2\pi}{T}\left(t-\frac{T}{4}\right)\right)\\ &= \cos\left(\frac{2\pi t}{T}-\frac{\pi}{2}\right)\\ &= \cos\left(\omega t-\frac{\pi}{2}\right) = \sin(\omega t).\end{aligned}$$

Note that physical meaning of phase shift is time advance/delay, and 360° phase shift implies that a periodic signal is time shifted to its original form (one period time shifting).

Contrary to sinusoidal functions, sinusoidal sequences are not always periodic and, therefore, a sinusoidal sequence may never time shift to the original input sequence. In addition to that, a periodic cosine sequence may not time shift to sine sequence. In fact, cosine and sine sequences time shift each other if and only if the fundamental period N_0 of those sinusoidal sequences is a multiple of 4.

Example 3.1 Derive the most abstract analytic expressions of sinusoidal sequences shown below.



Solution

$$x_1[n] = \cos(2\pi n/12) = \cos(\pi n/6),$$

$$y_1[n] = x_1[n-3] = \cos(\pi(n-3)/6) = \cos(\pi n/6 - \pi/2)$$
$$= \sin(\pi n/6),$$

$$x_2[n] = \cos(2\pi n/10) = \cos(\pi n/5),$$

$$y_2[n] = x_2[n+2] = \cos(\pi(n+2)/5) = \cos(\pi n/5 + 2\pi/5).$$

3.2.2 Time Reversal

Time reversal is expressed as

$$y(t) = x(-t) \quad \text{(continuous-time)}, \tag{3.7}$$
$$y[n] = x[-n] \quad \text{(discrete-time)}. \tag{3.8}$$

Time reversed signal is obtained as a reflection of input signal about the vertical axis ($t = 0$ or $n = 0$).

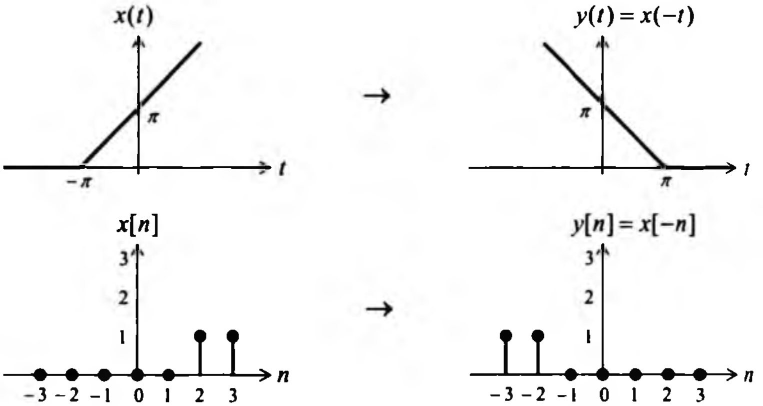


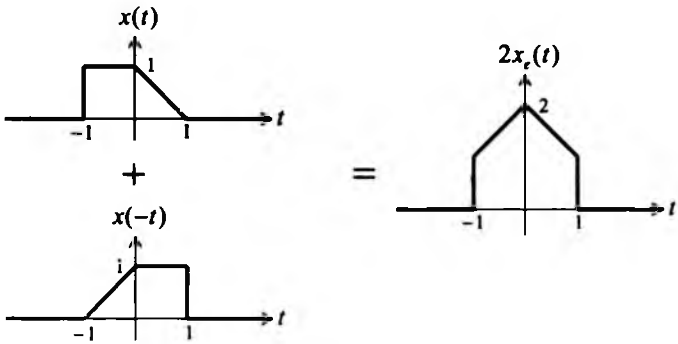
Figure 3.7: Examples of time reversal operations

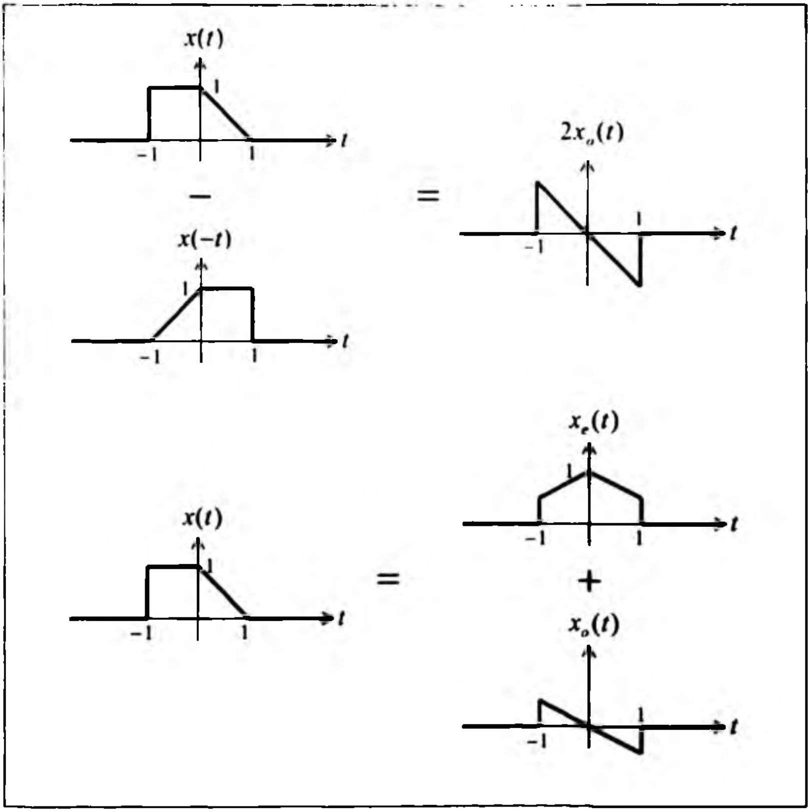
Example 3.2 Sketch even and odd parts of the following signal:

$$x(t) = u(t+1)\,u(-t) + u(t)\,r(-t+1).$$

Hint: Refer Example 1.4 for the shape of $x(t)$.

Solution





3.2.3 Time Scaling

Time scaling is expressed as

$$y(t) = x(at) \quad \text{(continuous-time)}, \tag{3.9}$$

$$y[n] = x[an] \quad \text{(discrete-time)}, \tag{3.10}$$

where a is a constant. The original signal is time compressed with $|a| > 1$ and time expanded with $|a| < 1$; a negative value of a invokes time reversal along with the time compression or expansion.

Figure 3.8 shows that for continuous-time signals, time compression can be always compensated by time expansion, and vice versa. In other words, one may perform time scaling operations without any loss of information. It is, however, not the case for discrete-time signals. Figure 3.9

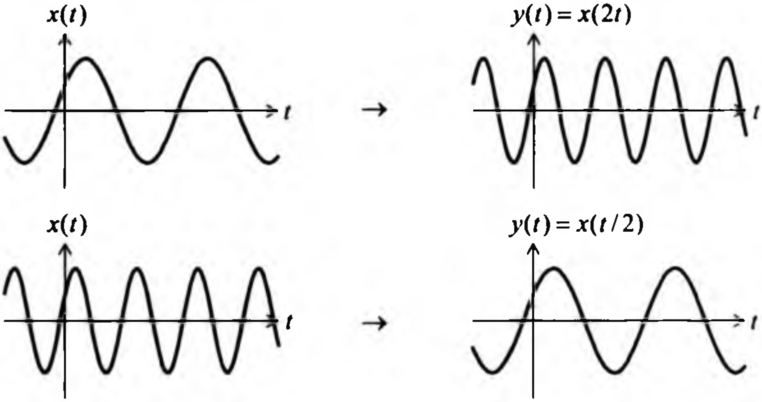


Figure 3.8: Examples of time scaling operations on continuous-time signal. Time expansion after time compression restores the original signal.

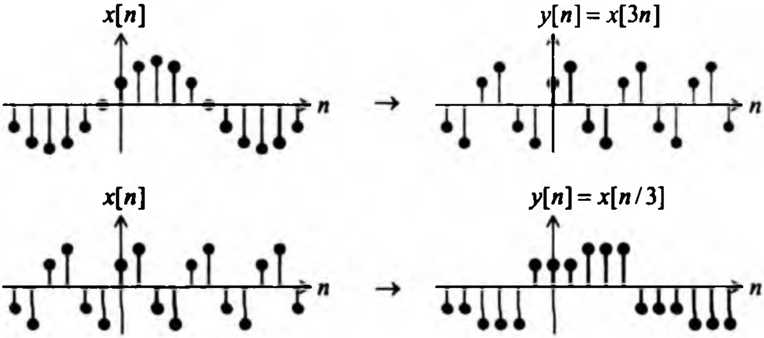


Figure 3.9: Examples of time scaling operations on discrete-time signal. Time expansion after time compression does not restore the original signal.

demonstrates that time compression of discrete-time signals may cause data loss.

3.2.4 Combined Operations on Continuous-Time Signals

Time operations are generally expressed as

$$y(t) = x(at + b).$$

This simple expression should be carefully interpreted, because the expression implicitly demands one to perform time shifting prior to time reversal or scaling. Consider performing time scaling (with factor a) prior to time shifting (with factor b). These two sequential operations are expressed as follows:

$$w(t) = x(at),$$
$$y(t) = w(t + b).$$

And the final expression of the output signal should be

$$y(t) = x(a(t + b)) = x(at + ab) \neq x(at + b).$$

On the other hand, performing time shifting (with factor b) ahead of time scaling (with factor a) is expressed as follows:

$$w(t) = x(t + b),$$
$$y(t) = w(at).$$

And the final expression of the output signal is

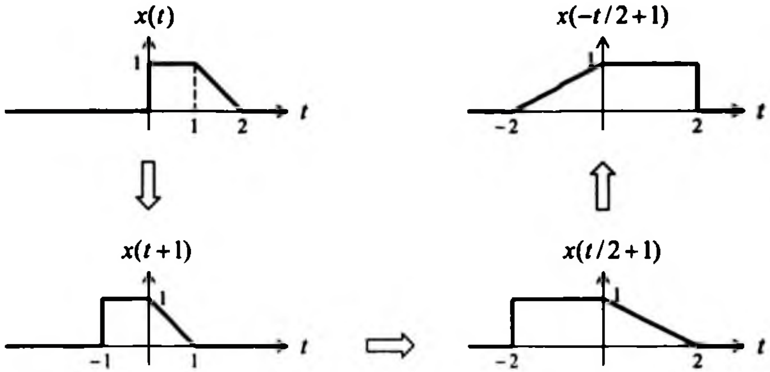
$$y(t) = x(at + b).$$

Example 3.3 Consider the following signal:

$$x(t) = u(t)\,u(-t + 1) + u(t - 1)\,r(-t + 2),$$

and sketch $x(-t/2 + 1)$. Hint: Refer Problem 1.5 for the shape of $x(t)$.

Solution

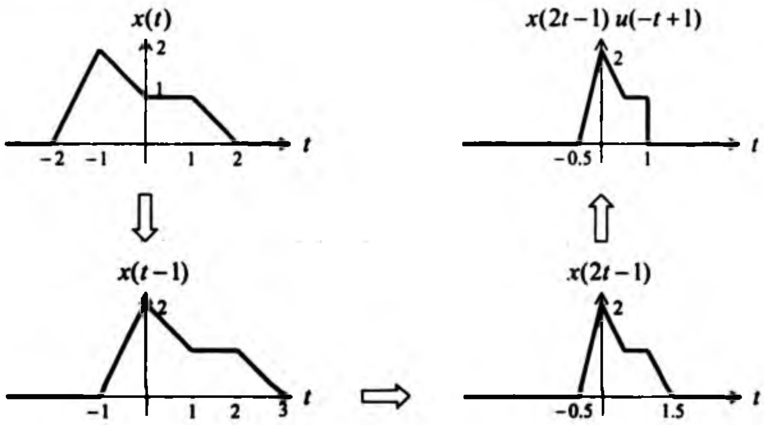


Example 3.4 Consider the following signal:

$$x(t) = 2r(t+2)\,u(-t-1) + u(t+1)\,u(-t)\,r(-t+1) \\ + u(t)\,u(-t+1) + u(t-1)\,r(-t+2),$$

and sketch $x(2t-1)\,u(-t+1)$. Hint: Refer Problem 1.6 for the shape of $x(t)$.

Solution

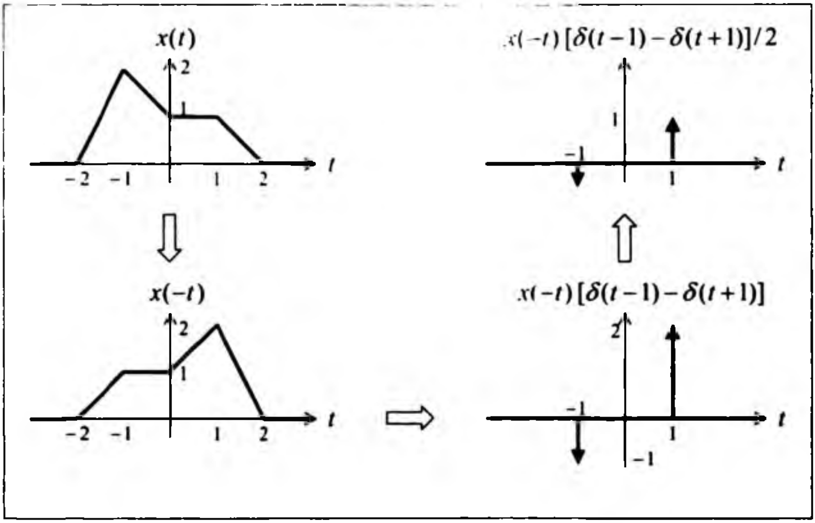


Example 3.5 Consider the following signal:

$$x(t) = 2r(t+2)\,u(-t-1) + u(t+1)\,u(-t)\,r(-t+1) \\ + u(t)\,u(-t+1) + u(t-1)\,r(-t+2),$$

and sketch $x(-t)\,[\delta(t-1) - \delta(t+1)]/2$. Hint: Refer Problem 1.6 for the shape of $x(t)$.

Solution



3.2.5 Combined Operations on Discrete-Time Signals

Sequential approach

We have discussed that while performing sequential operations on continuous-time signals, one needs to proceed time shifting prior to time reversal or scaling. Same principle works with discrete-time signals.

Consider the following operation with the sinusoidal sequence $x[n]$ in Figure 3.10:

$$y[n] = x[-2n - 2]$$

We separate the above operation into the following two sequential opera-

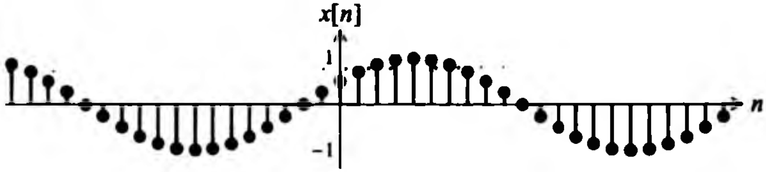


Figure 3.10: A sinusoidal time sequence

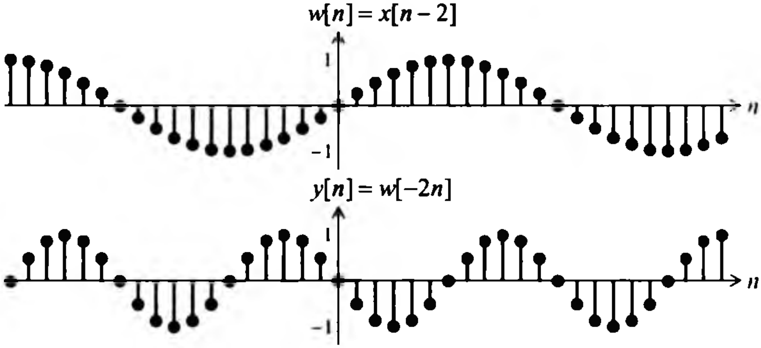


Figure 3.11: A graphical representation of sequential operations

tions:

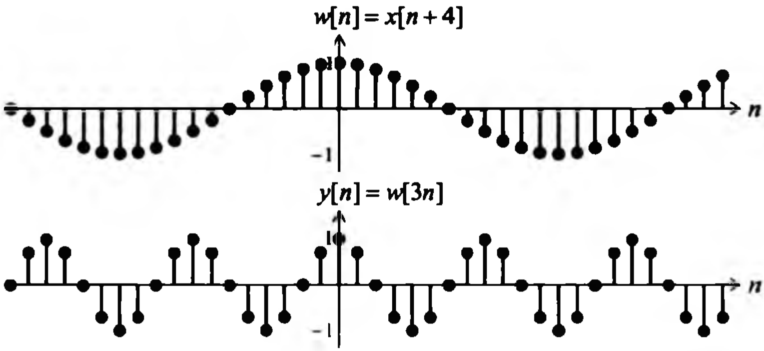
$$w[n] = x[n-2],$$
$$y[n] = w[-2n] = x[-2n-2].$$

Figure 3.11 shows the graphical representation of the sequential operations.

Example 3.6 Consider $x[n]$ in Figure 3.10, and sketch $y[n]$ that is given as

$$y[n] = x[3n+4].$$

Solution



Point matching approach

Sequential approach is not the only possible approach for discrete-time signals. If $x[n]$ is nonzero for a finite range of n, we may evaluate every possible $y[n]$. Consider once again the combined time operation that we have dealt with:

$$y[n] = x[-2n-2].$$

The above expression enables one to evaluate $y[n]$ as follows:

$$y[0] = x[-2],\ y[1] = x[-4],\ y[2] = x[-6],\ y[3] = x[-8],\ \cdots .$$

Figure 3.12 illustrates the above process. Note that Figures 3.11 and 3.12 exhibit the identical output signal $y[n]$.

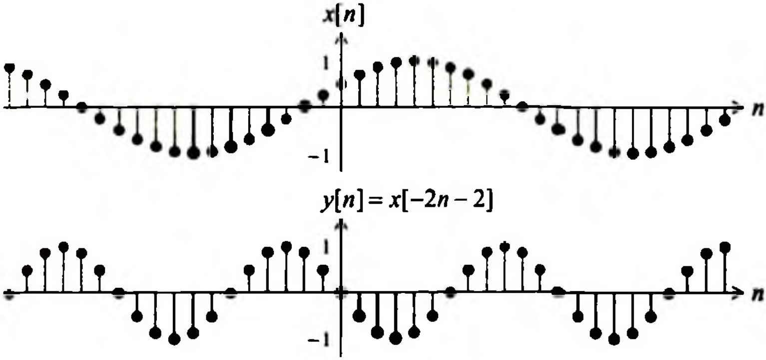


Figure 3.12: Point matching approach

Algebraic approach

We can occasionally rely on an algebraic approach as well. To do that, we first need the analytic expression of $x[n]$. Consider $x[n]$ in Figure 3.12. We note that the fundamental period N_0 is 24 and cosine sequence has time delayed with $b = -4$. We thus write that

$$x[n] = \cos(2\pi(n-4)/24) = \cos(\pi n/12 - \pi/3)$$

and derive the analytic expression of $y[n]$ in Figure 3.12 as

$$\begin{aligned} y[n] &= x[-2n-2] = \cos(\pi(-2n-2)/12 - \pi/3) = \cos(-\pi n/6 - \pi/2) \\ &= \cos(\pi n/6 + \pi/2) = -\sin(\pi n/6). \end{aligned}$$

Example 3.7 Consider $x[n]$ in Figure 3.12. Derive the most abstract expression of the following signal:

$$y[n] = x[3n+4],$$

and sketch the signal.

Solution

$$x[n] = \cos(\pi n/12 - \pi/3),$$
$$y[n] = x[3n+4] = \cos(\pi(3n+4)/12 - \pi/3) = \cos(\pi n/4).$$

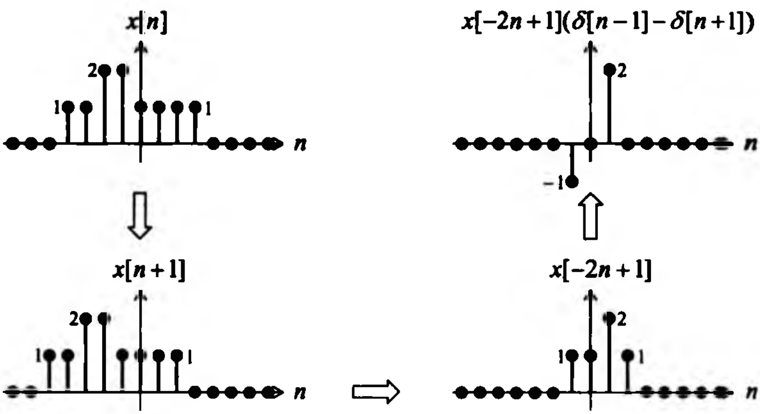
The graphical representation of $y[n]$ can be found in Example 3.6.

Example 3.8 Consider the following signal:

$$x[n] = u[n+4]\,u[-n+3] + \delta[n+1] + \delta[n+2],$$

and sketch $x[-2n+1]\,(\delta[n-1] - \delta[n+1])$.

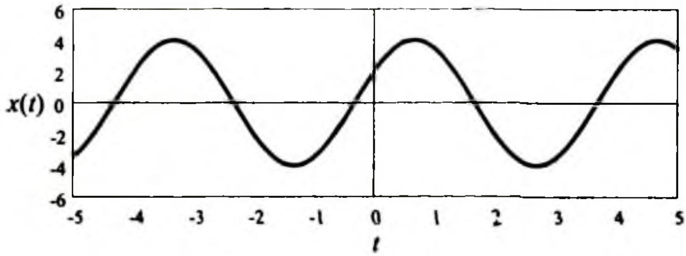
Solution



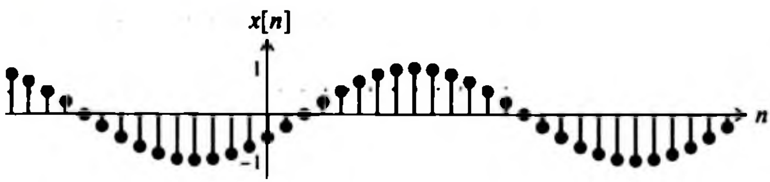
PROBLEMS

Problem 3.1 What is the analytic expression of the sinusoidal function $x(t)$?

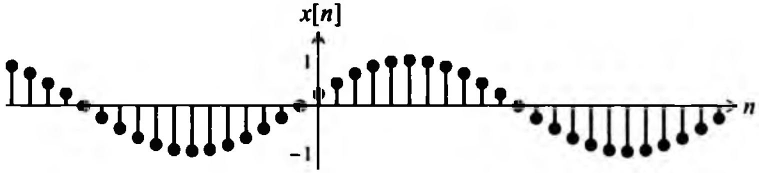
Problem 3.2 What is the analytic expression of the sinusoidal function $x(t)$?



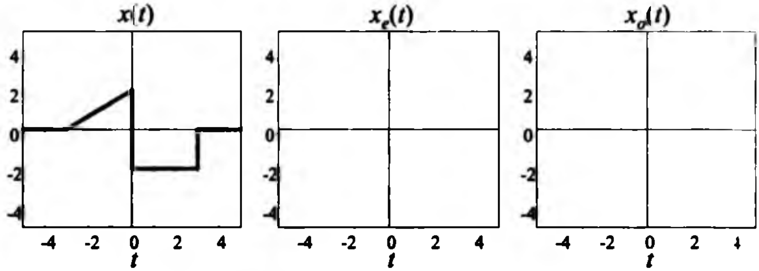
Problem 3.3 What is the algebraic expression of the sinusoidal sequence $x[n]$? What is the most abstract algebraic expression of $y[n] = x[3n-4]$?



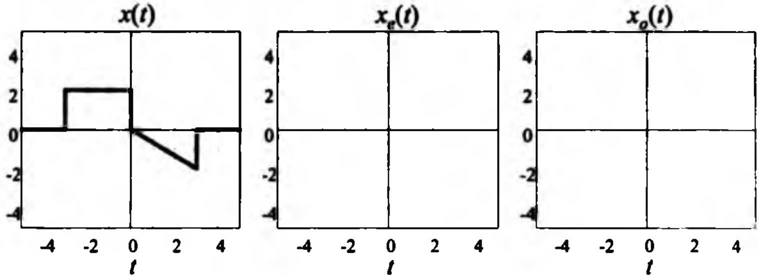
Problem 3.4 What is the algebraic expression of the sinusoidal sequence $x[n]$? What is the most abstract algebraic expression of $y[n] = x[2n+5]$?



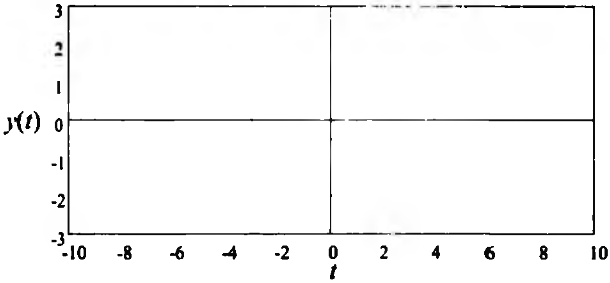
Problem 3.5 Sketch the even part $x_e(t)$ and odd part $x_o(t)$ of the time function $x(t)$.



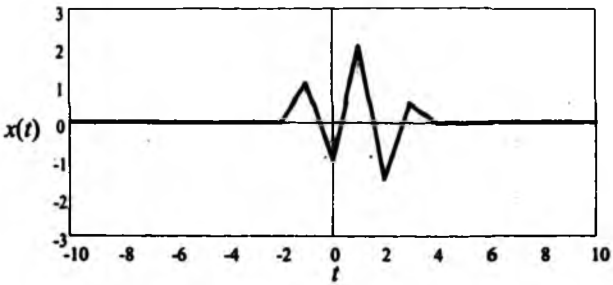
Problem 3.6 Sketch the even part $x_e(t)$ and odd part $x_o(t)$ of the time function $x(t)$.

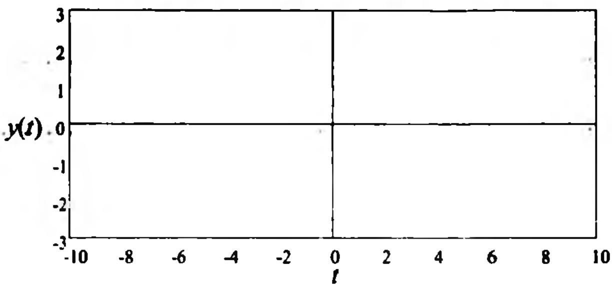


Problem 3.7 Sketch $y(t) = x(t/2 - 1)$.

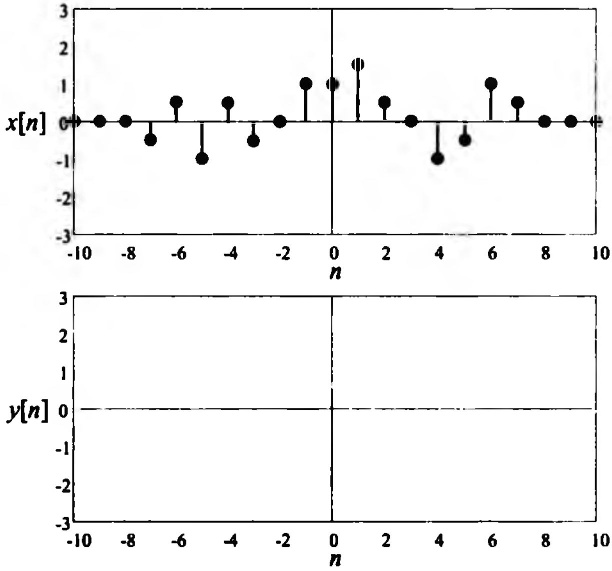


Problem 3.8 Sketch $y(t) = x(-t/2 + 1)$.

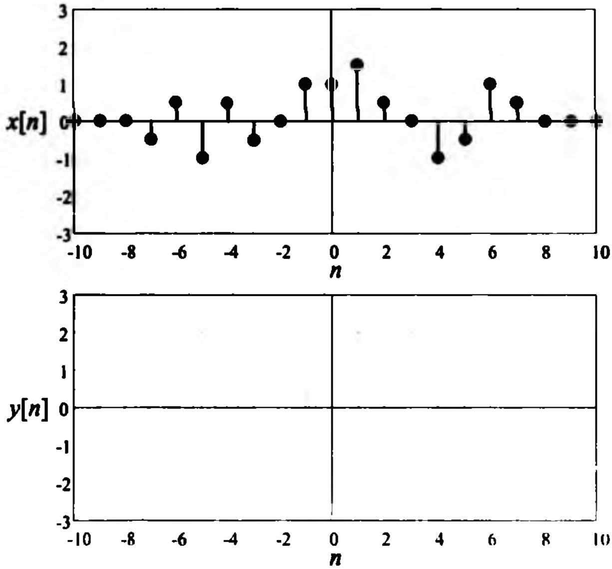




Problem 3.9 Sketch $y[n] = -2x[3n+2] + 1$.



Problem 3.10 Sketch $y[n] = x[2n-1] + 1$.



Chapter 4

UNDERSTANDING SYSTEMS

We have argued that systems may operate on signals in a variety of different ways. And the argument has been focused on signals instead of systems. We now focus more on systems themselves. A system may be regarded as a mathematical model of a physical process that relates input signals to output signals. Although a system may have many input and output signals, we focus our attention on the single-input single-output case.

4.1 CONTINUOUS-TIME AND DISCRETE-TIME SYSTEMS

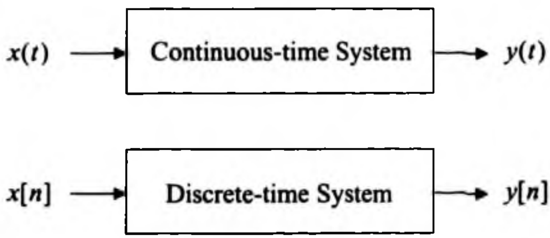


Figure 4.1: Continuous-time and discrete-time systems

A system is continuous-time if the input and output signals are time functions. It is discrete-time if the input and output signals are time sequences. In a *continuous-time system*, time is measured continuously, and the system is usually described by a differential equation, algebraic equation, polynomial equation, or integral equation, etc. For a *discrete-time system*, time is defined only at discrete instances and the system is described by a difference equation or any other way the input-output property of the system may be specified.

RC circuit are good examples of continuous-time systems. Figure 4.2 illustrates an RC circuit that has a *resistor* and *capacitor* whose *resistance* and *capacitance* are R and C, respectively. Considering the battery voltage as the input $x(t)$ to the system and the capacitor voltage as the output $y(t)$, we describe the property of the system via the following differential

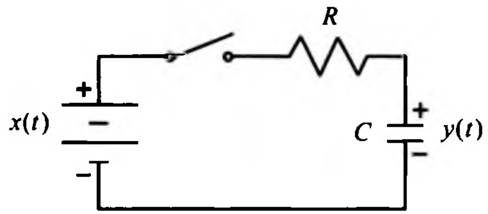


Figure 4.2: An RC circuit as a continuous-time system. $x(t)$ represents input battery voltage and $y(t)$ represents output voltage at the capacitor.

equation (Alexander and Sadiku 2016; Gilibisco and Monk 2016):

$$RC\frac{dy(t)}{dt} + y(t) = x(t). \tag{4.1}$$

As detailed in Chapter 5, one can analytically solve expression 4.1 and derive an input-output relationship shown in Figure 4.3.

A good example of discrete-time systems is a savings account. Suppose you visit a bank, open a savings account, and deposit a certain amount of money. You, from then on, use the account: one day you deposit more money, another day you withdraw your money. Note that while using the savings account, the balance of the account changes on a daily basis (i.e., without a new transaction, the balance remains the same throughout a day). Denoting the fixed value of the daily interest rate as ρ, we express the balance as follows:

$$y[n] = y[n-1] + \rho y[n-1] + x[n],$$

which states that the balance of the previous day ($y[n-1]$), interest that comes from the balance of the previous day ($\rho y[n-1]$), and any new

Figure 4.3: A continuous-time input to the RC circuit in Figure 4.2 yields a continuous-time output.

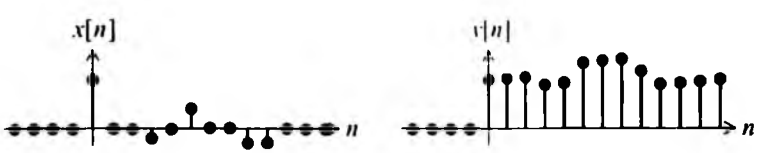


Figure 4.4: A savings account as a discrete-time system. $x[n]$ represents daily deposit or withdrawal and $y[n]$ represents daily balance of the savings account.

transaction ($x[n]$) altogether constitute the new balance. Rearranging terms of the above expression, we derive the difference equation that characterizes the savings account as

$$y[n] - (1 + \rho)y[n - 1] = x[n]. \tag{4.2}$$

Figure 4.4 illustrates the input-output relationship of the savings account.

4.2 LINEAR AND NONLINEAR SYSTEMS

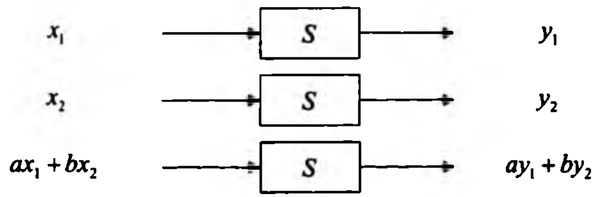


Figure 4.5: Concept of linear system

A *linear system* is one that guarantees a linear relationship between input and output signals. Suppose there has been two experiments with a system; input signals x_1 and x_2 have yielded output signals y_1 and y_2, respectively. Knowing the system is linear, one can safely argue that with a linear combination of x_1 and x_2 as an input (i.e., $ax_1 + bx_2$), we would get the same linear combination of y_1 and y_2 as the output (i.e., $ay_1 + by_2$).

Linearity is a nice property that ensures one to safely predict an output of a system. Nonlinearity, on the other hand, is troublesome, because, with a nonlinear system, it is generally hard to predict an output. Shock waves, for example, have to be considered nonlinear phenomena. The RC circuit in Figure 4.2 can safely be regarded as a linear system as far as the resistance R or capacitance C do not vary significantly with the input $x(t)$

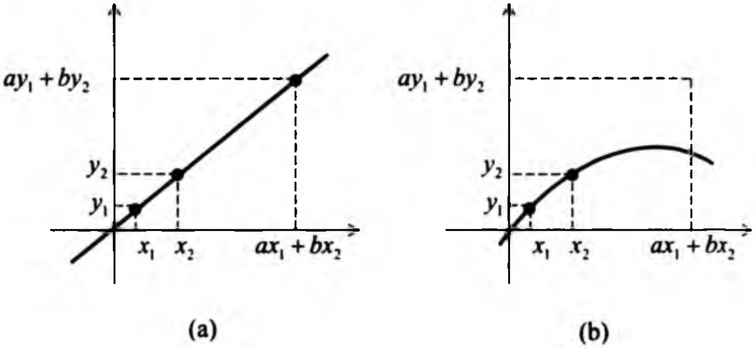


Figure 4.6: An analogy of linearity (a) and nonlinearity (b)

or output $y(t)$. Hearing an announcement that the bank would change the interest rate according to how much money remains in the savings account, we have to modify expression 4.2 into an appropriate nonlinear equation.

Example 4.1 A continuous-time system is described by the following expression:

$$y(t) = x^2(t).$$

Determine whether the system is linear or nonlinear.

Solution

We assume $y_1 = x_1^2$ and $y_2 = x_2^2$. We then write $x_3 = ax_1 + bx_2$, $y_3 = ay_1 + by_2$, and verify if $y_3 \stackrel{?}{=} x_3^2$.

LHS: $y_3 = ax_1^2 + bx_2^2$.

RHS: $x_3^2 = (ax_1 + bx_2)^2 = a^2x_1^2 + b^2x_2^2 + 2ab\, x_1 x_2$.

We see that $y_3 \neq x_3^2$, and thus the system is nonlinear.

Example 4.2 A continuous-time system is described by the following expression:

$$y(t) = \cos(3t)\,x(t).$$

Determine whether the system is linear or nonlinear.

Solution

We assume $y_1 = \cos(3t)\,x_1$ and $y_2 = \cos(3t)\,x_2$. We then write $x_3 = ax_1 + bx_2$, $y_3 = ay_1 + by_2$, and verify if $y_3 \stackrel{?}{=} \cos(3t)\,x_3$.

LHS: $y_3 = a\cos(3t)\,x_1 + b\cos(3t)\,x_2$.

RHS: $\cos(3t)\,x_3 = \cos(3t)\,(ax_1 + bx_2)$.

We see that $y_3 = \cos(3t)\,x_3$, and thus the system is linear.

Example 4.3 A continuous-time system is described by the following expression (R and C are constants):

$$RC\frac{dy(t)}{dt} + y(t) = x(t).$$

Determine whether the system is linear or nonlinear.

Solution

We assume $RC\,y_1' + y_1 = x_1$ and $RC\,y_2' + y_2 = x_2$. We then write $x_3 = ax_1 + bx_2$, $y_3 = ay_1 + by_2$, and verify if $RC\,y_3' + y_3 \stackrel{?}{=} x_3$.

LHS:
$$\begin{aligned} RC\,y_3' + y_3 &= RC\,(ay_1 + by_2)' + (ay_1 + by_2) \\ &= aRC\,y_1' + bRC\,y_2' + ay_1 + by_2 \\ &= a(RC\,y_1' + y_1) + b(RC\,y_2' + y_2) = ax_1 + bx_2 \end{aligned}$$

RHS: $x_3 = ax_1 + bx_2$.

We see that $RC\,y_3' + y_3 = x_3$, and thus the system is linear.

4.3 TIME-INVARIANT AND TIME-VARYING SYSTEMS

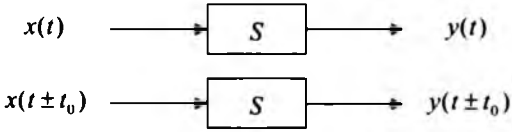


Figure 4.7: Concept of time-invariant system

A *time-invariant system* is one that maintains a consistent input-output relationship and satisfies an expectation that with an input-output result today, we would get the same input-output result tomorrow. The RC circuit in Figure 4.2 can safely be regarded as a time-invariant system as far as the resistance R and capacitance C are time-independent constants. Our savings account is a time-varying system if the interest rate ρ varies with time.

A quick way of diagnosing the time-invariance of a system is to consider two cases: (1) let the system work and then let time go, and (2) let time go and then let the system work. Having an identical result, we regard the system time-invariant. Otherwise, the system is a time-varying system. Figure 4.8 demonstrates that a time scaling system is NOT time-invariant. The system demonstrated in Figure 4.8 is expressed as

$$y(t) = x(2t).$$

We first assume an input signal $x(t)$, proceed the time compression and let time go (i.e., time shifting). The result is depicted as $y_1(t)$. Secondly, we time shift the input signal ahead of time compression. The result is illustrated as $y_2(t)$. We clearly observe that $y_2(t)$ differs from $y_1(t)$ and conclude that the system is a time-varying system.

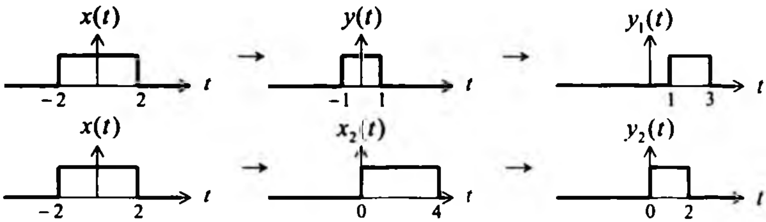


Figure 4.8: An example of time-varying system

Example 4.4 A continuous-time system is described by the following expression:

$$y(t) = \sin(x(t)).$$

Show that the system is time-invariant.

Solution

To prove the time-invariance of the system, we have to use the general representation of input signal $x(t)$.

$$x(t) \rightarrow y(t) = \sin(x(t)) \rightarrow y_1(t) = y(t \pm t_0) = \sin(x(t \pm t_0)),$$
$$x(t) \rightarrow x_2(t) = x(t \pm t_0) \rightarrow y_2(t) = \sin(x_2(t)) = \sin(x(t \pm t_0)).$$

We see $y_1(t) = y_2(t)$, and thus the system is time-invariant.

Example 4.5 A discrete-time system is described by the following expression:

$$y[n] = n\,x[n].$$

Show that the system is time-varying.

Solution

To prove the time-variance of the system, it is sufficient to show a counterexample. We consider $x[n] = \delta[n]$.

$$\begin{aligned} x[n] = \delta[n] &\rightarrow y[n] = n\,x[n] = n\,\delta[n] = 0 \\ &\rightarrow y_1[n] = y[n-1] = 0, \\ x[n] = \delta[n] &\rightarrow x_2[n] = x[n-1] = \delta[n-1] \\ &\rightarrow y_2[n] = n\,x_2[n] = n\,\delta[n-1] = \delta[n-1]. \end{aligned}$$

We see $y_1[n] \neq y_2[n]$, and thus the system is time-varying.

4.4 LINEAR TIME-INVARIANT SYSTEMS

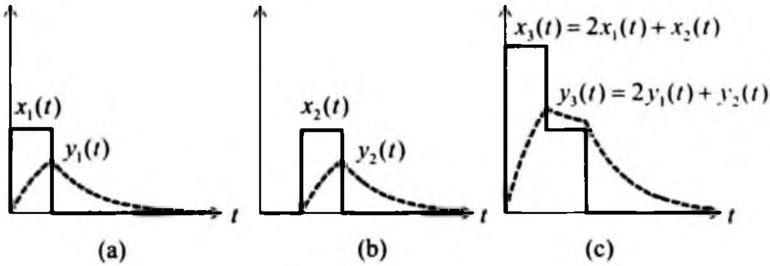


Figure 4.9: An LTI system and the principle of superposition

Linearity and time-invariance altogether substantiate the concept of *linear time-invariant* (LTI) system. As the name itself implies, an LTI system is one that is both linear and time-invariant. LTI systems are of our great interest because many physical systems can be safely considered linear and time-invariant. On the top of that, the *principle of superposition* is applicable to an LTI system.

Figure 4.9 (a) shows the input-output relation of the *RC circuit* depicted in Figure 4.2. We know we may safely regard the RC circuit as an LTI system. Time-invariance of the system allows one to establish the input-output relation in Figure 4.9 (b). And a linear combination of the input-output relations in Figure 4.9 (a) and 4.9 (b) yields the input-output relation in Figure 4.9 (c). Figure 4.9 well demonstrates how convenient it is to handle an LTI system. More detailed explanation about LTI system is provided in Chapter 5.

4.5 OTHER CLASSIFICATIONS OF SYSTEMS

4.5.1 BIBO Stable and Unstable Systems

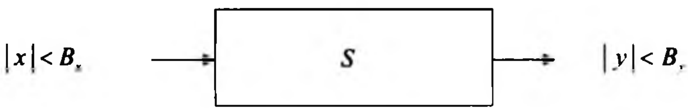


Figure 4.10: Concept of BIBO stable system

A *BIBO stable system* means that bounded-input to the system leads to bounded-output. Figure 4.10 shows the concept of bounded-input and bounded-output. Moving average systems are BIBO stable systems. Consider, for example, a discrete-time system that is described as

$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1]).$$

It is evident that bounded-input sequence $x[n]$ always leads to bounded-output $y[n]$.

An example of BIBO unstable system, on the other hand, is an *accumulator* system (Figure 4.11) that is expressed as

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{n-1} x[k] + x[n] = y[n-1] + x[n].$$

Assuming $x[n] = u[n]$, we derive $y[n] = r[n+1]$ (expression 1.13). In other words, bounded input ($|u[n]| \leq 1$) leads to unbounded output (indefinitely increasing ramp sequence). The accumulator is thus a BIBO unstable system.

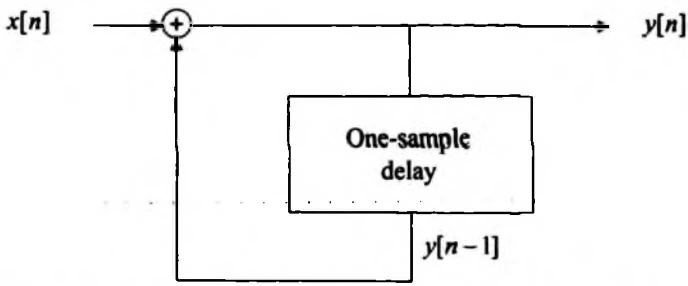


Figure 4.11: Block diagram of an accumulator system

4.5.2 Causal and Non-causal Systems

A *causal system* is one whose present response does not depend on future input. In other words, only past or present input values influence present output. Every physical system is causal, and causality is an essential constraint for time systems. It is known that an ideal filter is non-causal and is not physically realizable. It should be noted, however, that causality is not an essential constraint in applications where the independent variable is not time, such as image processing.

Examples of causal systems include

$$y[n] = (x[n])^2,$$
$$y[n] = x[n] + x[n-1],$$

whereas the following expressions represent non-causal systems:

$$y[n] = x[n+1] - x[n],$$
$$y[n] = x[1-n].$$

4.5.3 Systems With and Without Memory

A *memoryless system* is one in which present output depends only on present input; it does not depend on past or future input. When an output of a system depends on past input, the system is said to have a memory. A system with a memory is also called a *dynamic system*, whereas a memoryless system is called a *static system*.

An example of memoryless system is

$$y[n] = (x[n])^2.$$

Resistors are also considered to be memoryless:

$$y(t) = Rx(t),$$

where R, $x(t)$, and $y(t)$ represent the resistance of a resistor, input current to the resistor, and output voltage difference around the resistor, respectively.

Capacitors are, on the other hand, considered to have memory:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau,$$

where C, $x(t)$, and $y(t)$ represent the capacitance of a capacitor, input current to the conductor, and output voltage difference around the capacitor. The *accumulator* system in Figure 4.11 is another example of system with memory. The following expressions also represent systems that have memory:

$$y[n] = x[n] + x[n-1],$$
$$y[n] = x[n+1] - x[n],$$
$$y[n] = x[1-n].$$

4.6 INTERCONNECTED SYSTEMS

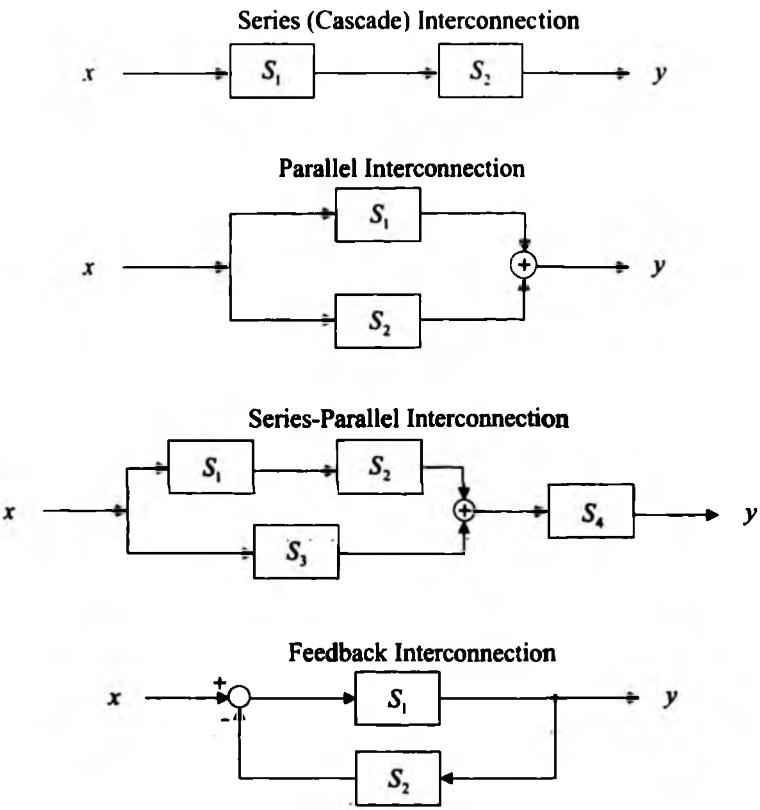


Figure 4.12: Block diagrams of several interconnected systems

Systems may be connected to form a larger system. In such a case, systems that constitute the larger system are called subsystems of the larger *interconnected system*. Depending on the way subsystems are connected, a variety of different interconnected systems can be realized. Figure 4.12 shows examples of interconnected systems.

Example 4.6 Consider the parallel interconnection in Figure 4.12. The subsystems of the interconnected system are described as follows:

$$S_1 \; : \; y_1[n] = 2x_1[n] + 4x_1[n-1],$$

$$S_2 \; : \; y_2[n] = x_2[n-2] + \frac{1}{2}x_2[n-3].$$

What then is the relation between the input $x[n]$ and output $y[n]$ sequences of the interconnected system?

Solution

$$\begin{aligned} y[n] &= y_1[n] + y_2[n] \\ &= 2x_1[n] + 4x_1[n-1] + x_2[n-2] + \frac{1}{2}x_2[n-3]. \end{aligned}$$

Note that $x[n] = x_1[n] = x_2[n]$. The final form of the output sequence thus becomes

$$y[n] = 2x[n] + 4x[n-1] + x[n-2] + \frac{1}{2}x[n-3].$$

Example 4.7 Consider the series interconnection in Figure 4.12. The subsystems of the interconnected system are described as follows:

$$S_1 \; : \; y_1[n] = 2x_1[n] + 4x_1[n-1],$$

$$S_2 \; : \; y_2[n] = x_2[n-2] + \frac{1}{2}x_2[n-3].$$

What then is the relation between the input $x[n]$ and output $y[n]$ sequences of the interconnected system?

Solution

$$y[n] = y_2[n] = x_2[n-2] + \frac{1}{2}x_2[n-3]$$

$$= y_1[n-2] + \frac{1}{2}y_1[n-3]$$

$$= (2x_1[n-2] + 4x_1[n-3]) + \frac{1}{2}(2x_1[n-3] + 4x_1[n-4])$$

$$= 2x_1[n-2] + 5x_1[n-3] + 2x_1[n-4]$$

$$= 2x[n-2] + 5x[n-3] + 2x[n-4].$$

Example 4.8 Consider the series interconnection in Figure 4.12. The subsystems of the interconnected system are described as follows:

$$S_1 \quad : \quad y_1[n] = x_1[n-2] + \frac{1}{2}x_1[n-3]$$

$$S_2 \quad : \quad y_2[n] = 2x_2[n] + 4x_2[n-1].$$

What then is the relation between the input $x[n]$ and output $y[n]$ sequences of the interconnected system? Compare the result with the result of the previous example and discuss why.

Solution

$$y[n] = y_2[n] = 2x_2[n] + 4x_2[n-1]$$

$$= 2y_1[n] + 4y_1[n-1]$$

$$= 2(x_1[n-2] + \frac{1}{2}x_1[n-3]) + 4(x_1[n-3] + \frac{1}{2}x_1[n-4])$$

$$= 2x_1[n-2] + 5x_1[n-3] + 2x_1[n-4]$$

$$= 2x[n-2] + 5x[n-3] + 2x[n-4].$$

The result is identical to the result of the previous example. The reason why we encounter the same result is because the two subsystems are LTI systems. Having a nonlinear or time-varying subsystem, we would face two different results for the two examples.

PROBLEMS

Problem 4.1 Show that a system described by the following differential equation is nonlinear:

$$\frac{dy(t)}{dt} + 1 = x(t).$$

Problem 4.2 Convince yourself that a system described by the following expression is time-invariant:

$$y[n] = x^2[n].$$

Problem 4.3 Which of the following differential equations represents linear and time-invariant systems? Choose one.

a. $\dfrac{d^2y(t)}{dt^2} - \dfrac{dy(t)}{dt} + y(t) = x(t)$

b. $\dfrac{d^2y(t)}{dt^2} + ty^2(t) = x(t)$

c. $\dfrac{d^2y(t)}{dt^2} + t^2y(t) = x(t)$

d. $\dfrac{d^2y(t)}{dt^2} + y(t)\dfrac{dy(t)}{dt} = x(t)$

Problem 4.4 Which of the following differential equations represents nonlinear and time-varying systems? Choose one.

a. $\dfrac{d^2y(t)}{dt^2} - \dfrac{dy(t)}{dt} + y(t) = x(t)$

b. $\dfrac{d^2y(t)}{dt^2} + ty^2(t) = x(t)$

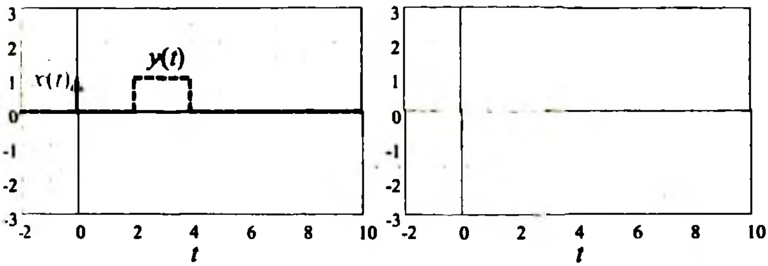
c. $\dfrac{d^2y(t)}{dt^2} + t^2y(t) = x(t)$

d. $\dfrac{d^2y(t)}{dt^2} + y(t)\dfrac{dy(t)}{dt} = x(t)$

Problem 4.5 Consider an LTI system that has the input-output relation shown in the left panel below. Suppose one does an experiment with the following input function:

$$x_1(t) = \delta(t-1) + \delta(t-2) - \delta(t-3).$$

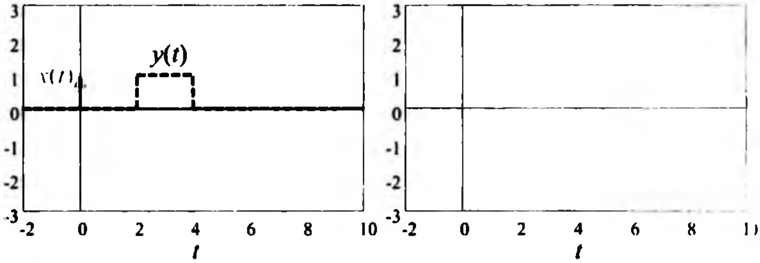
Sketch the output $y_1(t)$ one would get from the experiment.



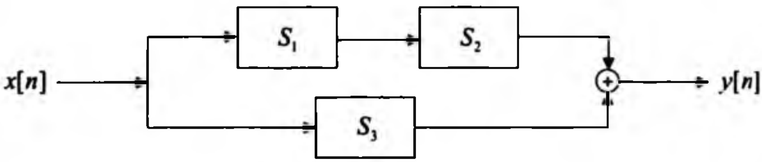
Problem 4.6 Consider an LTI system that has the input-output relation shown in the left panel below. Suppose one does an experiment with the following input function:

$$x_2(t) = \delta(t-1) - \delta(t-2) + \delta(t-3).$$

Sketch the output $y_2(t)$ one would get from the experiment.



Problem 4.7 Consider the following interconnected system:



The subsystems of the interconnected system are described as

$$
\begin{aligned}
S_1 &: \; y_1[n] = x_1[n-1], \\
S_2 &: \; y_2[n] = x_2[n-1], \\
S_3 &: \; y_3[n] = x_3[n-1].
\end{aligned}
$$

What is the relation between the input $x[n]$ and output $y[n]$ sequences of the interconnected system ?

Problem 4.8 Repeat Problem 4.7 with the following subsystems:

$$
\begin{aligned}
S_1 &: \; y_1[n] = x_1[n-1], \\
S_2 &: \; y_2[n] = x_2[n+1], \\
S_3 &: \; y_3[n] = x_3[n-2].
\end{aligned}
$$

Problem 4.9 Repeat Problem 4.7 with the following subsystems:

$$
\begin{aligned}
S_1 &: \; y_1[n] = x_1[n-1] - x_1[n-2], \\
S_2 &: \; y_2[n] = x_2[n-1] - x_2[n-2], \\
S_3 &: \; y_3[n] = x_3[n-1] - x_3[n-2].
\end{aligned}
$$

Problem 4.10 Repeat Problem 4.7 with the following subsystems:

$$
\begin{aligned}
S_1 &: \; y_1[n] = x_1[n] - x_1[n-1], \\
S_2 &: \; y_2[n] = x_2[n] - x_2[n-1], \\
S_3 &: \; y_3[n] = x_3[n] - x_3[n-1].
\end{aligned}
$$

Chapter 5

CONTINUOUS-TIME CONVOLUTION

We have argued that a system is a collection of devices that transforms input signals into output signals. The behavior of a system can be mathematically described either in the time domain or in the frequency domain. In Chapter 5, we introduce a technique called *convolution*, which is a basic tool for understanding systems in the time domain. A series of time domain approaches are possible, but we only concentrate on discussing convolution. Readers are encouraged to refer to Allen and Mills (2004) for a more comprehensive discussion about time domain approaches.

We first focus on the mathematical aspect of convolution and then learn how to apply it for assessing responses from linear time-invariant (LTI) systems. The reason we limit our discussion on LTI systems is twofold. First, no general procedures exist for non-LTI systems. Second, several physical systems can be modeled as LTI systems. Moreover, LTI systems can be analyzed in great detail because standard procedures are already available.

5.1 CONVOLUTION INTEGRAL

5.1.1 Introduction to Convolution Integral

The *convolution integral* or *superposition integral* of two functions $x(t)$ and $h(t)$ is defined as

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau. \tag{5.1}$$

Note that the asterisk symbol ($*$) is widely used to denote convolution integrals. Note also that the Greek letter τ is a popular choice for representing the dummy variable of the time integration. The integration in expression 5.1 can be regarded as a process that has following steps:

1. Folding or taking mirror image of $h(\tau)$ to obtain $h(-\tau)$,

2. Choosing a value of t to calculate the convolution integral,

3. Shifting $h(-\tau)$ according to the value of t to obtain $h(t-\tau)$,

4. Multiplying $x(\tau)$ and $h(t-\tau)$,

5. Integrating $x(\tau)\,h(t-\tau)$ over τ to evaluate the convolution for the specific value of t,

6. Repeating from step 2 to consider every possible t values.

Sketching $h(t-\tau)$ is, among the above steps, the key to the convolution process.

To better substantiate the idea of the convolution integral, consider a simple case that one convolve the unit step function with itself:

$$y(t) = u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)\,u(t-\tau)\,d\tau.$$

Figure 5.1 illustrates several intermediate steps necessary for the evaluation of the above convolution integral. It is evident from the graphical

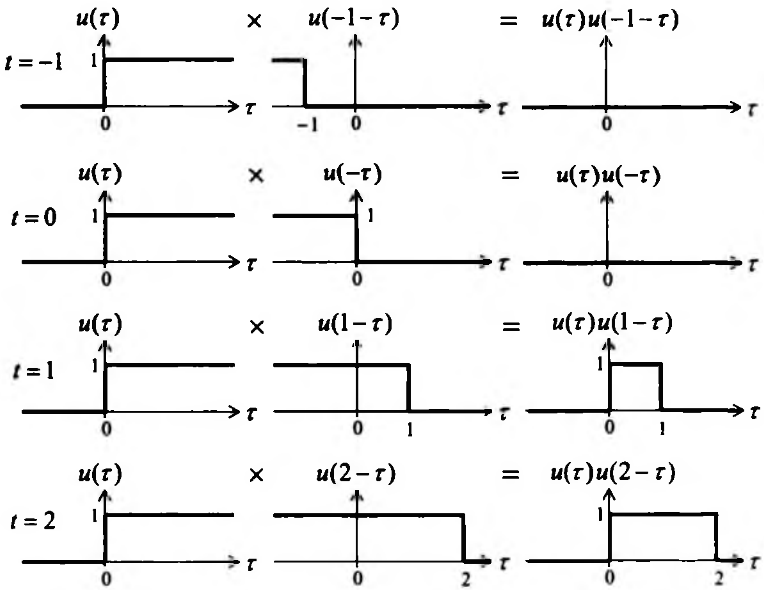


Figure 5.1: Graphical analysis for the evaluation of $u(t) * u(t)$

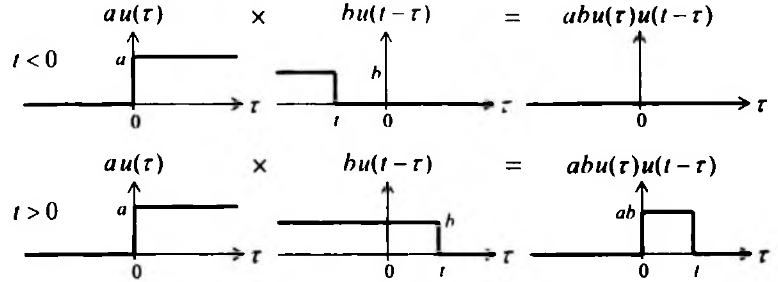


Figure 5.2: Graphical analysis for the evaluation of $[a\,u(t)] * [b\,u(t)]$

interpretation that

$$\begin{aligned}
y(-1) &= \int_{-\infty}^{\infty} u(\tau)\,u(-1-\tau)\,d\tau = 0, \\
y(0) &= \int_{-\infty}^{\infty} u(\tau)\,u(-\tau)\,d\tau = 0, \\
y(1) &= \int_{-\infty}^{\infty} u(\tau)\,u(1-\tau)\,d\tau = 1, \\
y(2) &= \int_{-\infty}^{\infty} u(\tau)\,u(2-\tau)\,d\tau = 2,
\end{aligned}$$

and one can deduce that

$$u(t) * u(t) = t\,u(t) = r(t). \tag{5.2}$$

Let us now repeat the above argument with a more general notation. Figure 5.2 exhibits that $[a\,u(t)] * [b\,u(t)] = 0$ for $t < 0$. For $t > 0$, on the other hand, the convolution becomes

$$[a\,u(t)] * [b\,u(t)] = \int_0^t ab\,d\tau.$$

We thus establish the following expression:

$$[a\,u(t)] * [b\,u(t)] = \left[\int_0^t ab\,d\tau\right] u(t).$$

The above expression is, in fact, very interesting because it implies that convolution between functions that are zero for $t < 0$ can be evaluated by only integrating between 0 and t. Consider two functions

$$x(t) = f(t)\,u(t) \quad \text{and} \quad h(t) = g(t)\,u(t). \tag{5.3}$$

Convolution of the two functions then becomes

$$x(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)\, u(\tau)\, g(t-\tau)\, u(t-\tau)\, d\tau$$

$$= \left[\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right] u(t). \tag{5.4}$$

Note that for $0 < \tau < t$, $f(\tau) = x(\tau)$ and $g(t-\tau) = h(t-\tau)$. We can thus rewrite the above expression as

$$x(t) * h(t) = \left[\int_0^t x(\tau)\, h(t-\tau)\, d\tau\right] u(t).$$

Most physical experiments involve signals that are zero until the outset of those experiments. In other words, signals are frequently expressed in the form of expression 5.3, and, in such a case, the range of convolution integral reduces significantly.

Example 5.1 Derive $x(t) * h(t)$ with the following two functions:

$$x(t) = u(t),$$
$$h(t) = r(t) = t\, u(t).$$

Solution

$$f(t) = 1 \quad \text{and} \quad g(t) = t.$$

$$x(t) * h(t) = \left[\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right] u(t) = \left[\int_0^t (t-\tau)\, d\tau\right] u(t)$$

$$= \left[t \int_0^t d\tau - \int_0^t \tau d\tau\right] u(t) = \left[t^2 - \frac{t^2}{2}\right] u(t) = \frac{t^2}{2}\, u(t).$$

Example 5.2 Derive $x(t) * h(t)$ with the following two functions:

$$x(t) = r(t) = t\, u(t),$$
$$h(t) = u(t).$$

Solution

$$f(t) = t \quad \text{and} \quad g(t) = 1.$$

$$x(t) * h(t) = \left[\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right] u(t) = \left[\int_0^t \tau\, d\tau\right] u(t) = \frac{t^2}{2} u(t).$$

Examples 5.1 and 5.2 exemplify that convolution is commutative such that

$$x(t) * h(t) = h(t) * x(t), \tag{5.5}$$

or

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau. \tag{5.6}$$

They also demonstrate that a convolution integral can be easier by making good choice about which one to fold and shift. It is generally a good idea to pause for a while before the calculation and investigate which one among $x(t)$ and $h(t)$ has a simpler form. It is, of course, better to fold and shift the simpler expression.

Example 5.3 Derive the convolution of the following two functions:

$$x(t) = e^{-at} u(t),$$
$$h(t) = u(t),$$

for $a > 0$.

Solution

$$f(t) = e^{-at} \quad \text{and} \quad g(t) = 1.$$

$$x(t) * h(t) = \left[\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right] u(t) = \left[\int_0^t e^{-a\tau}\, d\tau\right] u(t)$$
$$= \frac{1}{a}(1 - e^{-at})\, u(t).$$

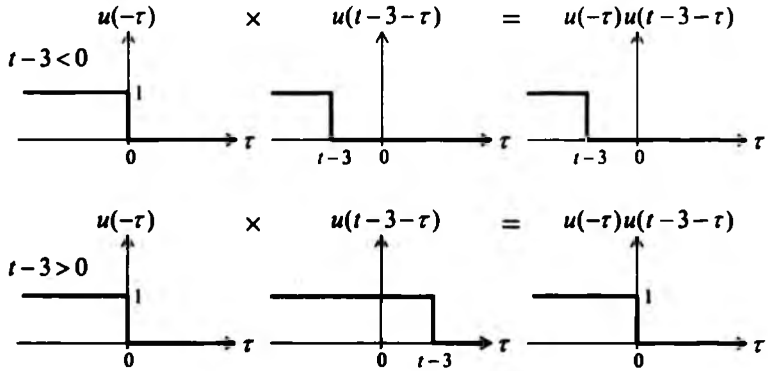
Example 5.4 Derive $x(t) * h(t)$ with the following two functions:

$$x(t) = e^{2t}\, u(-t),$$
$$h(t) = u(t-3).$$

Solution

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{2\tau}\, u(-\tau)\, u(t-3-\tau)\, d\tau.$$

The integration necessitates one to do the following graphical analysis:



Therefore, for $t < 3$,

$$x(t) * h(t) = \int_{-\infty}^{t-3} e^{2\tau}\, d\tau = \frac{1}{2}\, e^{2(t-3)},$$

and for $t > 3$,

$$x(t) * h(t) = \int_{-\infty}^{0} e^{2\tau}\, d\tau = \frac{1}{2}.$$

Combining the two cases, we write

$$x(t) * h(t) = \frac{1}{2}\, e^{2(t-3)}\, u(3-t) + \frac{1}{2}\, u(t-3).$$

5.1.2 Properties of Convolution Integral

We summarize several important properties of convolution integral as follows:

1. $x(t) * h(t) = h(t) * x(t)$ (commutative)
2. $x(t) * [g(t) + h(t)] = x(t) * g(t) + x(t) * h(t)$ (distributive)
3. $x(t) * [g(t) * h(t)] = [x(t) * g(t)] * h(t)$ (associative)
4. $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$
5. $x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau = \int_{-\infty}^{t} x(\tau)\,d\tau$
6. $\delta(t) * \delta(t) = \delta(t)$
7. $u(t) * u(t) = r(t)$

Another important property of convolution integral is the *width property* (Figure 5.3). If the duration of $x(t)$ and $h(t)$ are T_x and T_h, respectively, then the duration of their convolution is $T_x + T_h$.

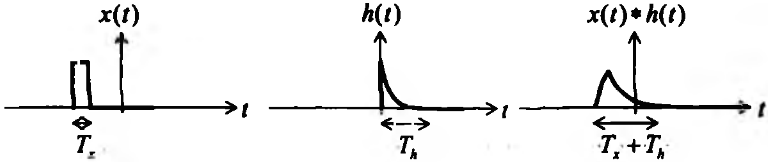


Figure 5.3: Width property of convolution integral

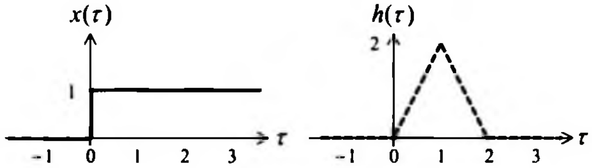
5.1.3 More Graphical Approaches

Example 5.5 Sketch $x(t) * h(t)$ with the following two functions:

$$x(t) = u(t),$$
$$h(t) = 2r(t)\,u(1-t) + 2r(2-t)\,u(t-1).$$

Solution

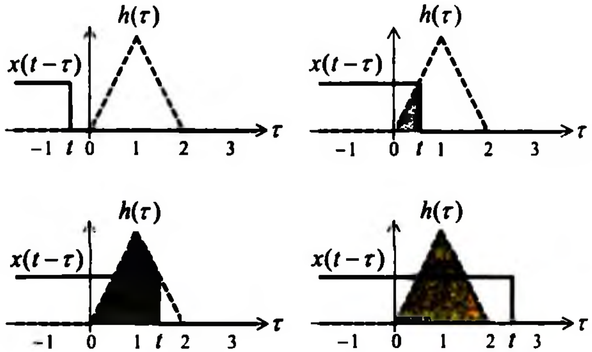
We first sketch $x(\tau)$ and $h(\tau)$.



Considering the complexity of $h(t)$, we fold and shift $x(t)$ such that

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau.$$

We then consider four different cases: $t < 0$, $0 < t < 1$, $1 < t < 2$, and $t > 2$.



It is evident that for $t < 0$,

$$x(t-\tau)\, h(\tau) = 0, \quad \text{and} \quad x(t) * h(t) = 0.$$

For $0 < t < 1$,

$$x(t-\tau)\, h(\tau) = \begin{cases} 0 & (\tau < 0), \\ 2\tau & (0 < \tau < t), \\ 0 & (t < \tau), \end{cases}$$

and

$$x(t) * h(t) = \int_0^t 2\tau \, d\tau = t^2.$$

For $1 < t < 2$,

$$x(t-\tau)\,h(\tau) = \begin{cases} 0 & (\tau < 0), \\ 2\tau & (0 < \tau < 1), \\ 4 - 2\tau & (1 < \tau < t), \\ 0 & (t < \tau), \end{cases}$$

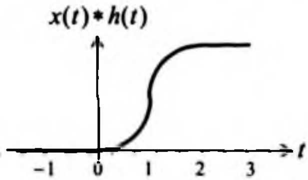
and

$$x(t) * h(t) = \int_0^1 2\tau \, d\tau + \int_1^t (4 - 2\tau)\, d\tau = -t^2 + 4t - 2.$$

And finally for $t > 2$,

$$x(t-\tau)\,h(\tau) = h(\tau) \quad \text{and} \quad x(t) * h(t) = 2.$$

Combining the four cases, we sketch the convolution result as follow.



Example 5.6 Sketch $x(t) * h(t)$ with the following two functions:

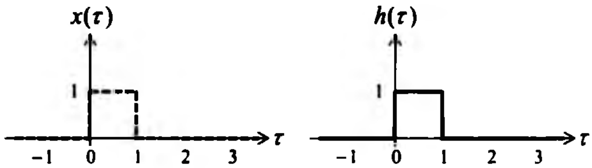
$$x(t) = u(t)\,u(1-t),$$
$$h(t) = u(t)\,u(1-t).$$

Solution

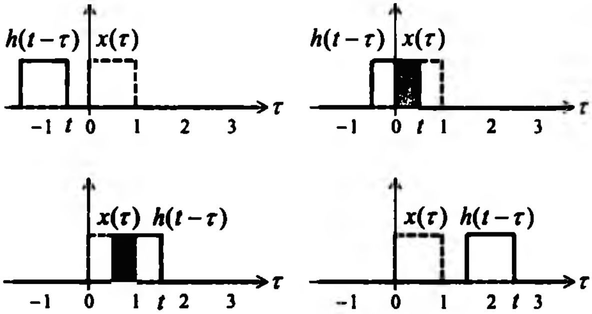
We first write

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau,$$

and sketch $x(\tau)$ and $h(\tau)$.



We then consider four different cases: $t < 0$, $0 < t < 1$, $1 < t < 2$, and $2 < t$.



It is evident that for $t < 0$,

$$x(\tau)\, h(t - \tau) = 0, \quad \text{and} \quad x(t) * h(t) = 0.$$

For $0 < t < 1$,

$$x(\tau)\, h(t - \tau) = \begin{cases} 0 & (\tau < 0), \\ 1 & (0 < \tau < t), \\ 0 & (t < \tau), \end{cases}$$

and

$$x(t) * h(t) = \int_0^t d\tau = t.$$

For $1 < t < 2$,

$$x(\tau)\,h(t-\tau) = \begin{cases} 0 & (\tau < t-1), \\ 1 & (t-1 < \tau < 1), \\ 0 & (1 < \tau). \end{cases}$$

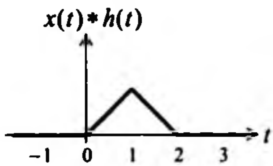
and

$$x(t) * h(t) = \int_{t-1}^{1} d\tau = 2 - t.$$

And finally for $2 < t$,

$$x(\tau)\,h(t-\tau) = 0 \quad \text{and} \quad x(t) * h(t) = 0.$$

Combining the four cases, we sketch the convolution result as follow.



Example 5.7 Derive $x(t) * h(t)$ with the following two functions:

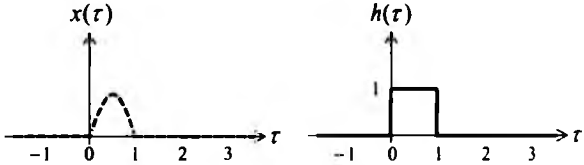
$$x(t) = u(t)\,u(1-t)\,\sin(\pi t),$$
$$h(t) = u(t)\,u(1-t).$$

Solution

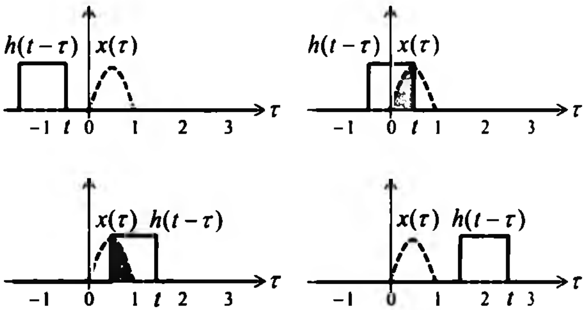
We first write

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau,$$

and sketch $x(\tau)$ and $h(\tau)$.



We then consider four different cases: $t < 0$, $0 < t < 1$, $1 < t < 2$, and $2 < t$.



It is evident that for $t < 0$,

$$x(\tau)\, h(t - \tau) = 0, \quad \text{and} \quad x(t) * h(t) = 0.$$

For $0 < t < 1$,

$$x(\tau)\, h(t - \tau) = \begin{cases} 0 & (\tau < 0), \\ \sin(\pi t) & (0 < \tau < t), \\ 0 & (t < \tau), \end{cases}$$

and

$$x(t) * h(t) = \int_0^t \sin(\pi t)\, d\tau = \frac{1 - \cos(\pi t)}{\pi}.$$

For $1 < t < 2$,

$$x(\tau)\, h(t - \tau) = \begin{cases} 0 & (\tau < t - 1), \\ \sin(\pi t) & (t - 1 < \tau < 1), \\ 0 & (1 < \tau), \end{cases}$$

and

$$x(t) * h(t) = \int_{t-1}^{1} \sin(\pi t)\, d\tau = \frac{\cos(\pi t - \pi) - \cos(\pi)}{\pi} = \frac{1 - \cos(\pi t)}{\pi}.$$

And finally for $2 < t$,

$$x(\tau)\, h(t - \tau) = 0 \quad \text{and} \quad x(t) * h(t) = 0.$$

Combining the four cases, we write the convolution result as follow:

$$x(t) * h(t) = \frac{1 - \cos(\pi t)}{\pi}\, u(t)\, u(2 - t).$$

5.2 IMPULSE RESPONSE AND CONVOLUTION

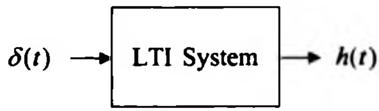


Figure 5.4: Concept of impulse response

Having discussed the mathematical aspect of convolution integral, we consider its significance for *linear time-invariant* (LTI) systems, which necessitate one to understand the concept of impulse response. The *impulse response* $h(t)$ is the response of an LTI system when the input is the unit impulse function $\delta(t)$ (Figure 5.4). Impulse response may be analyzed by analytic study or experimental measurements. And once we know the impulse response of an LTI system, we may say we understand the system. The meaning of understanding the system is that with the impulse response, we can predict the response of the system to any input signals via the convolution of the input signal and impulse response.

For the proof of the above argument, recall the *sifting property* of the unit impulse function shown in expression 1.6. After changing notations ($t \to \tau$, and $t_0 \to t$), changing sign of the argument of the delta function, extending the range of integration, we can rewrite expression 1.6 as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t - \tau)\, d\tau.$$

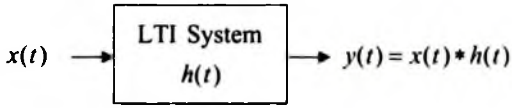


Figure 5.5: Impulse response and convolution

The above expression means that any input function $x(t)$ can be expressed in the form of convolution integral with the unit impulse function. And using the above expression as the input to an LTI system, we may express the output from the system as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

In other words, with an input function $x(t)$, the output from an LTI system is given as

$$y(t) = x(t) * h(t), \tag{5.7}$$

where $h(t)$ is the impulse response of the LTI system. Note that the linear and time-invariant nature of the system enables one to write expression 5.7. Figure 5.5 summarizes the argument as a block diagram.

5.2.1 Impulse Response and BIBO Stability

We have studied in Chapter 4 that for a *BIBO stable system*, bounded (finite) input signals always lead to bounded (finite) output signals. The BIBO stability of a system can be manifested in terms of the impulse response of the system. Following conventional notations of an input signal $x(t)$, impulse response $h(t)$, and output signal $y(t)$, we write

$$|y(t)| = |x(t) * h(t)| = \left| \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau \right| \le \int_{-\infty}^{\infty} |x(t-\tau)|\, |h(\tau)|\, d\tau.$$

Assuming that the input signal is bounded:

$$|x(t-\tau)| \le K,$$

with a positive contant K, the magnitude of the output should satisfy the following expression:

$$|y(t)| \le \int_{-\infty}^{\infty} K\, |h(\tau)|\, d\tau \le K \int_{-\infty}^{\infty} |h(\tau)|\, d\tau.$$

The above expression implies that the output signal $y(t)$ is bounded if the impulse response $h(t)$ satisfies the following condition:

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty. \tag{5.8}$$

In other words, a continuous-time system is BIBO stable if its impulse response is absolutely integrable.

5.2.2 Impulse Response and Causality

Utilizing the concept of the impulse response, one may argue that for a *causal system*, the impulse response should have no signal before the impulsive input at $t = 0$ such that

$$h(t) = 0 \quad (t < 0). \tag{5.9}$$

For causal systems, calculating convolution can be a simpler task such that

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau.$$

Incidentally, convolution integrals can be even simpler with an extra assumption that $x(t) = 0$ for $t < 0$. With that assumption, convolution integrals are expressed as

$$y(t) = x(t) * h(t) = \left[\int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau\right] u(t). \tag{5.10}$$

Example 5.8 Consider the following impulse responses:

1. $h(t) = e^{-3t}\, u(t-3)$
2. $h(t) = u(t)$
3. $h(t) = e^{t} u(-t)$
4. $h(t) = e^{2t}\, u(t+3)$

Determine the BIBO stability and causality of the systems each the above impulse responses represent.

Solution

1. Stable / Causal
2. Unstable / Causal
3. Stable / Noncausal
4. Unstable / Noncausal

5.2.3 Impulse Response of Interconnected Systems

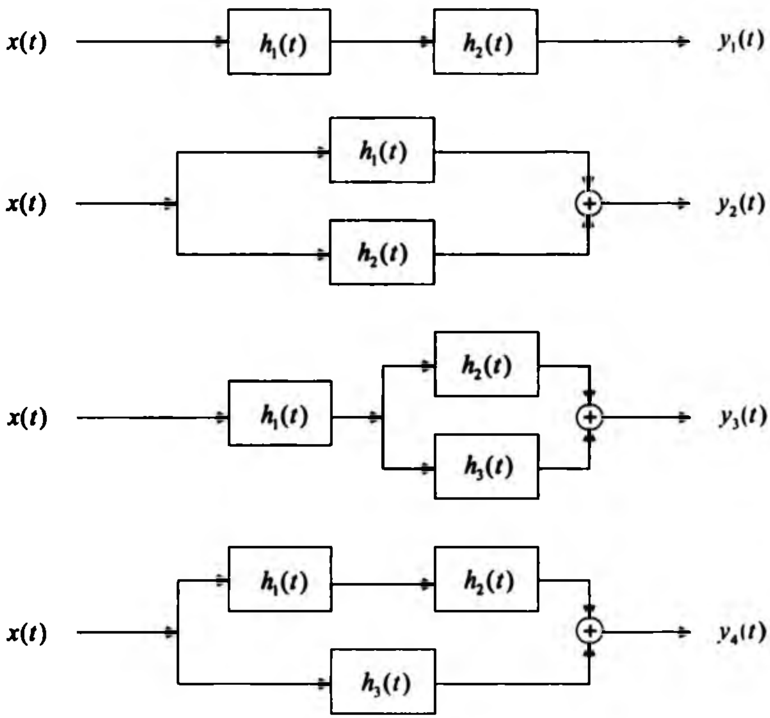


Figure 5.6: Interconnected convolution

Impulse response is a useful concept that can characterize *interconnected systems*. For example, the impulse response of an interconnected system can be described by combining or convolving impulse responses of its constituent subsystems. Figure 5.6 shows a list of interconnected systems whose input-output relations are express as follows:

$$y_1(t) = x(t) * [h_1(t) * h_2(t)],$$

$$y_2(t) = x(t) * [h_1(t) + h_2(t)],$$

$$y_3(t) = x(t) * h_1(t) * [h_2(t) + h_3(t)],$$

$$y_4(t) = x(t) * [h_1(t) * h_2(t) + h_3(t)].$$

5.3 A STUDY ABOUT RC CIRCUIT

In Chapter 4, we have considered an *RC circuit* (Figure 4.2) and discussed that the input battery voltage $x(t)$ and output capacitor voltage $y(t)$ are related by the following differential equation:

$$RC\frac{dy(t)}{dt} + y(t) = x(t). \qquad (5.11)$$

We have also discussed that the RC circuit is a good example of LTI systems. In this chapter, we consider the RC circuit again and study how the convolution integral works. The impulse response of the LTI system is

$$h(t) = \frac{1}{RC}e^{-t/RC}u(t). \qquad (5.12)$$

We will study in Chapter 10 that the above expression can be conveniently derived via frequency domain analysis. While staying within the time domain, on the other hand, it requires a more sophisticated mathematical approach to derive the impulse response. Readers may skip the derivation of expression 5.12 and focus on the working principle of the convolution integral.

5.3.1 Derivation of the Impulse Response

We rewrite expression 5.11 as

$$RC\frac{dh(t)}{dt} + h(t) = \delta(t). \qquad (5.13)$$

The system must be causal such that

$$h(t) = g(t)\, u(t).$$

The differential equation thus becomes

$$RC\,\frac{dg(t)}{dt}\,u(t) + RC\,g(t)\,\frac{du(t)}{dt} + g(t)\,u(t) = \delta(t).$$

Applying expression 1.9, we rewrite the above expression as

$$RC\,\frac{dg(t)}{dt}\,u(t) + RC\,g(t)\,\delta(t) + g(t)\,u(t) = \delta(t).$$

And the sifting property of the impulse function (expression 1.7) yields

$$RC\,\frac{dg(t)}{dt}\,u(t) + RC\,g(0)\,\delta(t) + g(t)\,u(t) = \delta(t).$$

We now decompose the above expression into the following two equations:

$$RC\,g(0)\,\delta(t) = \delta(t),$$

and

$$RC\,\frac{dg(t)}{dt}\,u(t) + g(t)\,u(t) = 0.$$

The first equation provides the following initial condition:

$$g(0) = \frac{1}{RC},$$

and the second equation becomes

$$RC\,\frac{dg(t)}{dt} + g(t) = 0.$$

It is well known that the above differential equation describes exponentially decaying phenomena and its solution has the following form: $g(t) = ae^{bt}$. Substituting the form into the above two equations, we identify that $a = 1/RC$, $b = -1/RC$, and

$$g(t) = \frac{1}{RC}\,e^{-t/RC}.$$

The final expression of $h(t)$ thus becomes

$$h(t) = \frac{1}{RC}\,e^{-t/RC}\,u(t).$$

It is evident from the above expression that the impulse response of the RC circuit suddenly increases at $t = 0$ and, from then on, exhibits exponential decrease.

5.3.2 RC Circuit and Convolution Integral

Knowing the impulse response of the RC circuit, we can consider any input functions and derive the output functions via the convolution integral. Consider the following input function to the RC circuit:

$$x(t) = u(t)\,u(t_0 - t),$$

with $t_0 > 0$. The output from the circuit is then expressed as

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$
$$= \int_{-\infty}^{\infty} [u(\tau)\,u(t_0-\tau)] \left[\frac{1}{RC} e^{-(t-\tau)/RC}\,u(t-\tau)\right] d\tau.$$

Applying expression 5.4, we rewrite the above expression as

$$y(t) = \frac{1}{RC}\left[\int_0^t u(t_0-\tau)\,e^{-(t-\tau)/RC}\,d\tau\right] u(t).$$

We now consider two different cases: $t < t_0$ and $t_0 < t$. Note that for $t < t_0$, the step function $u(t_0 - \tau)$ is always 1 within the integration range:

$$u(t_0 - \tau) = 1 \quad (0 < \tau < t).$$

For $t_0 < t$, on the other hand, the step function $u(t_0 - \tau)$ is not always 1 within the integration range:

$$u(t_0-\tau) = \begin{cases} 1 & (0 < \tau < t_0), \\ 0 & (t_0 < \tau < t). \end{cases}$$

For $t < t_0$, the output function thus becomes

$$y(t) = \frac{1}{RC}\left[\int_0^t e^{-(t-\tau)/RC}\,d\tau\right] u(t) = \frac{1}{RC} e^{-t/RC}\left[\int_0^t e^{\tau/RC}\,d\tau\right] u(t)$$
$$= e^{-t/RC}\left[e^{t/RC} - 1\right] u(t) = \left[1 - e^{-t/RC}\right] u(t),$$

and for $t_0 < t$,

$$y(t) = \frac{1}{RC}\left[\int_0^{t_0} e^{-(t-\tau)/RC}\,d\tau\right] u(t) = \frac{1}{RC} e^{-t/RC}\left[\int_0^{t_0} e^{\tau/RC}\,d\tau\right] u(t)$$
$$= e^{-t/RC}\left[e^{t_0/RC} - 1\right] u(t).$$

Combining the two cases, we finally write the expression of the output function from the RC circuit as

$$y(t) = \left[1-e^{-t/RC}\right] u(t)\, u(t_0-t) + e^{-t/RC}\left[e^{t_0/RC}-1\right] u(t-t_0). \quad (5.14)$$

The above expression shows that for $0 < t < t_0$, the output from the circuit increases with time, whereas, for $t > t_0$, the output exhibits exponential decrease as time goes on.

Figure 5.7 demonstrates how one may utilize expression 5.14 to evaluate different output signals from the RC circuit. The *RC time constant*, which is the multiplication of the resistance R and conductance C of the circuit, is 1 second. At first, the input battery is activated for 1 second ($t_0 = RC$), and we observe that the output capacitor voltage $y_1(t)$ increases upto about 63% of the input voltage. The capacitor voltage exhibits exponential decrease after the disconnection of the input battery. Secondly, we consider activating the input battery for 5 seconds ($t_0 = 5RC$). We observe that the output capacitor voltage $y_2(t)$ increases upto about 99% of the input voltage. We also clearly observe that the capacitor voltage increases fast at the beginning but the rate of increase quickly gets slower as time goes on.

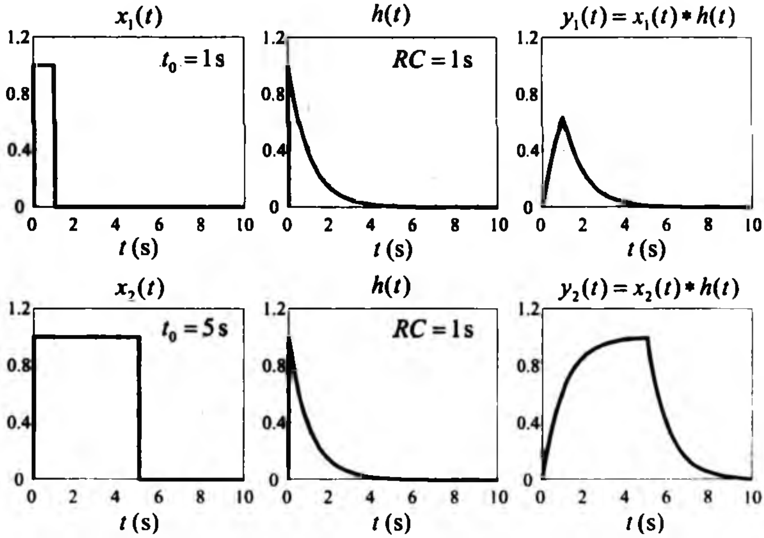


Figure 5.7: RC circuit and convolution integral

PROBLEMS

Problem 5.1 Calculate the convolution of the following two functions:

$$x(t) = r(t),$$
$$h(t) = r(t).$$

Problem 5.2 Calculate the convolution of the following two functions:

$$x(t) = u(t),$$
$$h(t) = t^2 u(t).$$

Problem 5.3 Calculate the convolution of the following two functions:

$$x(t) = \cos(\pi t)\, u(t),$$
$$h(t) = u(t).$$

Problem 5.4 Calculate the convolution of the following two functions:

$$x(t) = \sin(\pi t)\, u(t)\, u(2 - t),$$
$$h(t) = u(t).$$

Problem 5.5 Sketch the convolution of the following two functions.

$$x(t) = u(t)\, u(2 - t),$$
$$h(t) = u(t)\, u(4 - t).$$

Problem 5.6 Sketch the convolution of the following two functions:

$$x(t) = u(t - 1)\, u(5 - t),$$
$$h(t) = u(t - 2)\, u(6 - t).$$

Problem 5.7 Consider an LTI system whose impulse response is

$$h(t) = e^{-5t}\,u(2-t).$$

Which of the following correctly describes the LTI system ? Choose one.

a. Causal and BIBO stable

b. Noncausal but BIBO stable

c. Causal but BIBO unstable

d. Noncausal and BIBO unstable

Problem 5.8 Consider an LTI system whose impulse response is

$$h(t) = e^{-3t}\,u(t+2).$$

Which of the following correctly describes the LTI system ? Choose one.

a. Causal and BIBO stable

b. Noncausal but BIBO stable

c. Causal but BIBO unstable

d. Noncausal and BIBO unstable

Problem 5.9 Show that expression 5.12 does satisfy expression 5.13.

Problem 5.10 Show that expression 5.14 is continuous at $t = t_0$.

Chapter 6

DISCRETE-TIME CONVOLUTION

The concept of convolution integral has been discussed with respect to continuous-time systems. Similar concept exists with respect to discrete-time systems. In Chapter 6, we first introduce the concept of convolution sum and then discuss its application for analyzing linear time-invariant (LTI) systems. We also introduce the concept of deconvolution and demonstrate how to accomplish convolution / deconvolution via digital computers. The numerical work, which is based on a package called MATLAB, will soon demonstrate how efficient it is to use digital computer to perform convolution / deconvolution of not only discrete-time signals but also of continuous-time signals.

6.1 CONVOLUTION SUM

6.1.1 Introduction to Convolution Sum

The *convolution sum* or *superposition sum* of two sequences $x[n]$ and $h[n]$ is defined as

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]. \tag{6.1}$$

Note that the asterisk symbol ($*$) may represent both the continuous-time convolution and discrete-time convolution. Note also that the letter k is a popular choice for representing the dummy variable of the time summation. Similarly to convolution integral, the summation in expression 6.1 can be regarded as a process that has the following steps:

1. Folding or taking mirror image of $h[k]$ to obtain $h[-k]$,

2. Choosing a value of n to calculate the convolution sum,

3. Shifting $h[-k]$ according to the value of n to obtain $h[n-k]$,

4. Multiplying $x[k]$ and $h[n-k]$.

5. Summing $x[k]\,h[n-k]$ over k to evaluate the convolution for the specific value of n,

6. Repeating from step 2 to consider every possible n values.

Sketching $h[n-k]$ is, of course, the key to the convolution process.

As the first example of the convolution sum, consider a simple case that one convolve the unit step sequence with itself:

$$u[n] * u[n] = \sum_{k=-\infty}^{\infty} u[k]\,u[n-k].$$

The summation result of the above expression may differ depending on the value of n. Figure 6.1 illustrates that $u[n] * u[n] = 0$ for $n < 0$. On the other hand, the summation becomes

$$u[n] * u[n] = \sum_{k=0}^{n} 1,$$

for $n \geq 0$. We can thus establish the following expression:

$$\begin{aligned} u[n] * u[n] &= \sum_{k=-\infty}^{\infty} u[k]\,u[n-k] = \left(\sum_{k=0}^{n} 1\right) u[n] \\ &= (n+1)\,u[n] = r[n+1]. \end{aligned} \tag{6.2}$$

The above expression now leads one to consider convolving two time sequences that are zero for $n < 0$ as follows:

$$x[n] = f[n]\,u[n] \quad \text{and} \quad h[n] = g[n]\,u[n]. \tag{6.3}$$

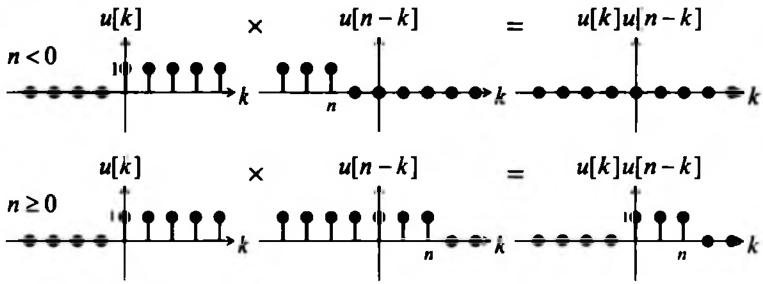


Figure 6.1: Graphical analysis for the evaluation of $u[n] * u[n]$

Convolution of the two sequences then becomes

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} f[k]\, u[k]\, g[n-k]\, u[n-k]$$
$$= \left(\sum_{k=0}^{n} f[k]\, g[n-k]\right) u[n]. \tag{6.4}$$

Note that for $0 \leq k \leq n$, $f[k] = x[k]$ and $g[n-k] = h[n-k]$. We can thus rewrite the above expression as

$$x[n] * h[n] = \left(\sum_{k=0}^{n} x[k]\, h[n-k]\right) u[n].$$

Expression 6.4 means that for signals that are zero until the outset of a physical experiment, the range of convolution sum reduces significantly.

Example 6.1 Derive $x[n] * h[n]$ with the following two sequences:

$$x[n] = u[n],$$
$$h[n] = r[n] = n\, u[n].$$

Hint: Refer to expression B.47.

Solution

$$f[n] = 1 \quad \text{and} \quad g[n] = n.$$

$$x[n] * h[n] = \left(\sum_{k=0}^{n} f[k]\, g[n-k]\right) u[n] = \left(\sum_{k=0}^{n} (n-k)\right) u[n]$$
$$= \left(\sum_{k=0}^{n} n - \sum_{k=0}^{n} k\right) u[n] = \left(\sum_{k=0}^{n} n - \sum_{k=1}^{n} k\right) u[n]$$
$$= \left(n\,(n+1) - \frac{n(n+1)}{2}\right) u[n] = \frac{n(n+1)}{2}\, u[n].$$

Example 6.2 Derive $x[n] * h[n]$ with the following two sequences:

$$x[n] = r[n] = n\,u[n],$$
$$h[n] = u[n].$$

Solution

$$f[n] = n \quad \text{and} \quad g[n] = 1.$$

$$x[n] * h[n] = \left(\sum_{k=0}^{n} f[k]\,g[n-k]\right) u[n] = \left(\sum_{k=0}^{n} k\right) u[n]$$
$$= \left(\sum_{k=1}^{n} k\right) u[n] = \frac{n(n+1)}{2}\,u[n].$$

Examples 6.1 and 6.2 exemplify that convolution is commutative such that

$$x[n] * h[n] = h[n] * x[n], \tag{6.5}$$

or

$$\sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k]. \tag{6.6}$$

It is therefore desirable to fold and shift the one that has a simpler expression among $x[n]$ and $h[n]$.

Example 6.3 Derive $x[n] * h[n]$ with the following two sequences:

$$x[n] = r[n] = n\,u[n],$$
$$h[n] = r[n] = n\,u[n].$$

Hint: Refer to expression B.48.

Solution

$$f[n] = n \quad \text{and} \quad g[n] = n.$$

$$\begin{aligned}
x[n] * h[n] &= \left(\sum_{k=0}^{n} f[k]\, g[n-k]\right) u[n] = \left(\sum_{k=0}^{n} k(n-k)\right) u[n] \\
&= \left(n \sum_{k=0}^{n} k - \sum_{k=0}^{n} k^2\right) u[n] = \left(n \sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2\right) u[n] \\
&= \left(n\, \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}\right) u[n] \\
&= \frac{n(n+1)(n-1)}{6}\, u[n].
\end{aligned}$$

Example 6.4 Derive $x[n] * h[n]$ with the following two sequences:

$$\begin{aligned}
x[n] &= a^n\, u[n], \\
h[n] &= u[n],
\end{aligned}$$

for $a \neq 1$. Hint: Refer to expression B.51.

Solution

$$f[n] = a^n \quad \text{and} \quad g[n] = 1.$$

$$\begin{aligned}
x[n] * h[n] &= \left(\sum_{k=0}^{n} f[k]\, g[n-k]\right) u[n] = \left(\sum_{k=0}^{n} a^k\right) u[n] \\
&= (1 + a + a^2 + \cdots + a^n)\, u[n] = \frac{a^{n+1} - 1}{a - 1}\, u[n].
\end{aligned}$$

Example 6.5 Derive $x[n] * h[n]$ with the following two sequences:

$$x[n] = (0.7)^n u[n],$$
$$h[n] = (0.2)^n u[n].$$

Solution

$$f[n] = (0.7)^n \quad \text{and} \quad g[n] = (0.2)^n.$$

$$\begin{aligned} x[n] * h[n] &= \left(\sum_{k=0}^{n} f[k]\, g[n-k]\right) u[n] \\ &= \left(\sum_{k=0}^{n} (0.7)^k (0.2)^{n-k}\right) u[n] \\ &= (0.2)^n \left(\sum_{k=0}^{n} (0.7/0.2)^k\right) u[n] \\ &= (0.2)^n \frac{(0.7/0.2)^{n+1} - 1}{0.7/0.2 - 1} u[n] \\ &= \frac{(0.7)^{n+1} - (0.2)^{n+1}}{0.5} u[n] \\ &= 2\{(0.7)^{n+1} - (0.2)^{n+1}\}\, u[n] \end{aligned}$$

6.1.2 Properties of Convolution Sum

We summarize several important properties of convolution sum as follows:

1. $x[n] * h[n] = h[n] * x[n]$ (commutative)
2. $x[n] * (g[n] + h[n]) = x[n] * g[n] + x[n] * h[n]$ (distributive)
3. $x[n] * (g[n] * h[n]) = (x[n] * g[n]) * h[n]$ (associative)

4. $x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] = x[n]$

5. $x[n] * u[n] = \sum_{k=-\infty}^{\infty} x[k]\,u[n-k] = \sum_{k=-\infty}^{n} x[k]$

6. $\delta[n] * \delta[n] = \delta[n]$

7. $u[n] * u[n] = r[n+1]$

Another important property of convolution sum is the *width property* (Figure 6.2). If $x[n]$ and $h[n]$ are N_x-points and N_h-points sequences, respectively, then the convolution of the two sequences yields $(N_x + N_h - 1)$-points sequence.

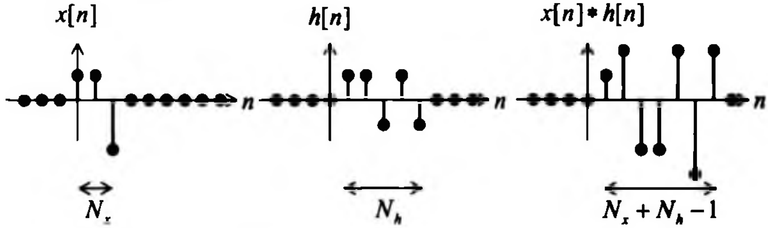


Figure 6.2: Width property of convolution sum

We finally introduce another useful property of convolution sum. The *shifting property* of convolution sum states that having the following convolution relation:

$$x[n] * h[n] = y[n],$$

one can associate the time shifting pattern of the individual sequences of the above expression as

$$x[n-n_1] * h[n-n_2] = y[n-n_1-n_2]. \tag{6.7}$$

In other words, once we know a convolution relationship between three sequences, we can freely time shift any of them and predict its influence. Applying the shifting property to the following convolution relation:

$$\delta[n] * \delta[n] = \delta[n],$$

we derive that

$$\delta[n - n_1] * \delta[n - n_2] = \delta[n - n_1 - n_2]. \tag{6.8}$$

Example 6.6 Derive $x[n] * h[n]$ with the following two sequences:

$$\begin{aligned} x[n] &= \delta[n] + 2\delta[n-1] - \delta[n-3], \\ h[n] &= 2\delta[n+1] + 2\delta[n-1]. \end{aligned}$$

Solution

$$\begin{aligned} x[n] * h[n] &= (\delta[n] + 2\delta[n-1] - \delta[n-3]) * (2\delta[n+1] + 2\delta[n-1]) \\ &= 2\delta[n] * \delta[n+1] + 2\delta[n] * \delta[n-1] \\ &\quad + 4\delta[n-1] * \delta[n+1] + 4\delta[n-1] * \delta[n-1] \\ &\quad - 2\delta[n-3] * \delta[n+1] - 2\delta[n-3] * \delta[n-1] \\ &= 2\delta[n+1] + 2\delta[n-1] + 4\delta[n] + 4\delta[n-2] \\ &\quad - 2\delta]n-2] - 2\delta[n-4] \\ &= 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4] \end{aligned}$$

6.1.3 Convolution Sum and Polynomial Multiplication

While solving Example 6.6, one might have noticed that convolving two delta sequences is a process similar to multiplying terms of polynomials such that

$$\delta[n-n_1] * \delta[n-n_2] = \delta[n-n_1-n_2] \quad \Longleftrightarrow \quad z^{-n_1}\, z^{-n_2} = z^{-n_1-n_2}.$$

The convolution sum has, in fact, a strong connection to *polynomial multiplication*. Recalling the *sifting property* of the unit impulse sequence (expression 1.18), we express two time sequences $x[n]$ and $h[n]$ as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k] \quad \text{and} \quad h[n] = \sum_{k=-\infty}^{\infty} h[k]\, \delta[n-k],$$

and, based on the above expressions, we also implement the following two polynomials:

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k} \quad \text{and} \quad H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}.$$

The convolution of the two time sequences $x[n] * h[n]$ can be, then, inferred from the multiplication of the two polynomials such that

$$y[n] = x[n] * h[n] \quad \Longleftrightarrow \quad Y(z) = X(z)\, H(z) = \sum_{k=-\infty}^{\infty} y[k]\, z^{-k}. \quad (6.9)$$

In other words, if we convert time sequences, $x[n]$ and $h[n]$, to polynomials of z (i.e., $X(z)$ and $H(z)$) and multiply the two polynomials, the coefficients of the multiplication result (i.e., $Y(z) = X(z)\, H(z)$) can be readily converted back to a time sequence that exactly matches with the convolution sum of the original time sequences: $y[n] = x[n] * h[n]$.

Example 6.7 Use the polynomial multiplication approach and derive $y[n] = x[n] * h[n]$ with the following two sequences:

$$\begin{aligned} x[n] &= \delta[n] + 2\delta[n-1] - \delta[n-3], \\ h[n] &= 2\delta[n+1] + 2\delta[n-1]. \end{aligned}$$

Solution

$$\begin{aligned} X(z) &= 1 + 2z^{-1} - z^{-3}, \\ H(z) &= 2z + 2z^{-1}. \end{aligned}$$

$$\begin{aligned} Y(z) = X(z)\, H(z) &= (1 + 2z^{-1} - z^{-3})(2z + 2z^{-1}) \\ &= (2z + 4 - 2z^{-2}) + (2z^{-1} + 4z^{-2} - 2z^{-4}) \\ &= 2z + 4 + 2z^{-1} + 2z^{-2} - 2z^{-4}. \end{aligned}$$

$$y[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4].$$

Example 6.8 Consider time sequences $x[n]$ and $y[n]$ in Example 6.7. Use the polynomial multiplication approach and derive $y[n] = x[n-2] * h[n-1]$.

Solution

$$\begin{aligned} Y(z) &= [z^{-2} X(z)]\,[z^{-1} H(z)] = z^{-3} X(z)\, H(z) \\ &= 2z^{-2} + 4^{-3} + 2z^{-4} + 2z^{-5} - 2z^{-7}. \end{aligned}$$

$$y[n] = 2\delta[n-2] + 4\delta[n-3] + 2\delta[n-4] + 2\delta[n-5] - 2\delta[n-7].$$

Examples 6.7 and 6.8 show that polynomial multiplication can be a convenient approach of convolution sum. Note that while transforming a time sequence into the polynomial form, it is customary to denote the name of the polynomial with the uppercase letter that corresponds to the name of the sequence. Note also that transforming a time sequence into a polynomial form is, in fact, strongly connected to a technique called the *z-Transform*. More about the technique will be presented in Chapter 13.

6.2 DECONVOLUTION

We have observed that the convolution sum is a process that can be cloned by polynomial multiplication. A question that naturally follows is whether there exists a process that may be cloned by *polynomial division*. In fact, there exists a process called *deconvolution* that polynomial division do clone.

Suppose one knows $x[n]$ and $y[n]$ of the following convolution relation:

$$y[n] = x[n] * h[n].$$

Assessing $h[n]$ from the above relation is then the deconvolution process. Note, however, that deconvolution is only meaningful for discrete-time signals, and no direct mathematical definition exists for continuous-time signals.

Example 6.9 Consider the following time sequences:

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3],$$
$$y[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4],$$

and derive $h[n]$ that satisfies the following convolution relation: $y[n] = x[n] * h[n]$.

Solution

$$X(z) = 1 + 2z^{-1} - z^{-3},$$
$$Y(z) = 2z + 4 + 2z^{-1} + 2z^{-2} - 2z^{-4},$$
$$H(z) = Y(z)/X(z).$$

$$\begin{array}{r|l}
 & 2z \quad + 2z^{-1} \\
\hline
1 + 2z^{-1} \quad - z^{-3} & 2z + 4 + 2z^{-1} + 2z^{-2} \quad - 2z^{-4} \\
 & 2z + 4 \qquad\quad - 2z^{-2} \\
\cline{2-2}
 & \qquad\quad 2z^{-1} + 4z^{-2} \quad - 2z^{-4} \\
 & \qquad\quad 2z^{-1} + 4z^{-2} \quad - 2z^{-4} \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad 0
\end{array}$$

$$H(z) = 2z + 2z^{-1},$$
$$h[n] = 2\delta[n+1] + 2\delta[n-1].$$

6.3 IMPULSE RESPONSE AND CONVOLUTION

We have seen that deconvolution is the process of obtaining one of the constituent sequences in the convolution sum. Deconvolution is also known as inverse filtering or system identification, because it is usually used for determining the *impulse response* of a *linear time-invariant* (LTI) system. The concept of the impulse response in discrete-time systems is basically identical to its counterpart in continuous-time systems; the impulse response $h[n]$ is the response of a discrete-time LTI system when

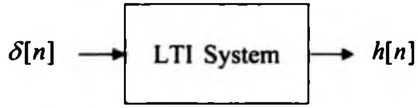


Figure 6.3: Concept of impulse response

the input to the system is the unit impulse sequence $\delta[n]$.

We have studied in Chapter 5 that the sifting property of the unit impulse function $\delta(t)$ enables one to associate the output $y(t)$ from an LTI system with the convolution integral of the input $x(t)$ and impulse response $h(t)$ of the LTI system. Same principle applies to discrete-time systems. The *sifting property* of the unit impulse sequence $\delta[n]$ states that any input sequence $x[n]$ can be expressed in the form of convolution sum with the impulse sequence:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k].$$

Upon taking the above expression as the input sequence, the linear and time-invariant nature of a discrete-time system yields the following output sequence:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k].$$

In other words, the convolution sum between the input sequence $x[n]$ and impulse response $h[n]$ yields the following output $y[n]$ from the LTI system:

$$y[n] = x[n] * h[n]. \tag{6.10}$$

Example 6.10 Consider a discrete-time LTI system that is characterized by the following impulse response:

$$h[n] = \delta[n-1] + \delta[n-2] - \delta[n-3] + \delta[n-4] - \delta[n-5].$$

Assume also the following input:

$$x[n] = \delta[n] + \delta[n-1] - 2\delta[n-2].$$

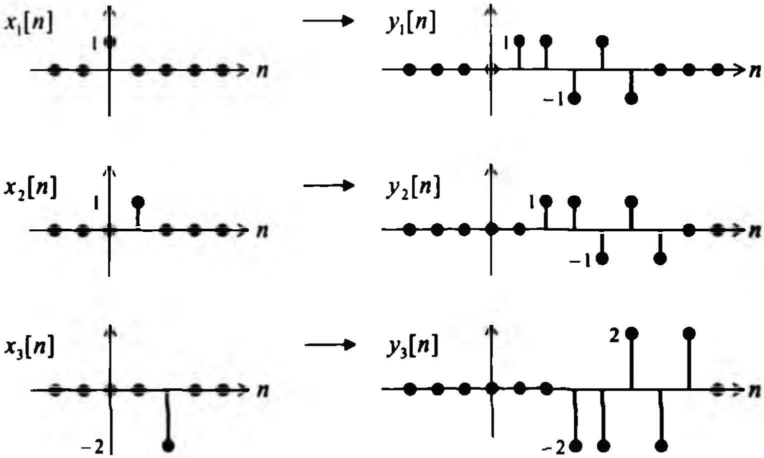
Do not use the convolution sum, only rely on the liner time-invariant nature of the system, and derive the output $y[n]$ from the system.

Solution

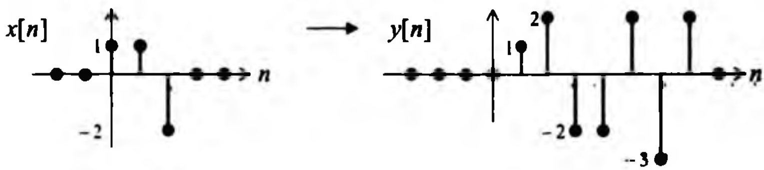
We decompose the input sequence into the combination of three impulse sequences as

$$x_1[n] = \delta[n], \quad x_2[n] = \delta[n-1], \quad \text{and} \quad x_3[n] = -2\delta[n-2].$$

Linear time-invariant nature of the system guarantees that each of the impulse sequences yields output sequences as follows:



The input $x[n]$ should thus accompany the following output $y[n]$:



The sequence $y[n]$ in Example 6.10 can be also derived by the polynomial multiplcation approach as follows:

$$X(z) = 1 + z^{-1} - 2z^{-2},$$

$$H(z) = z^{-1} + z^{-2} - z^{-3} + z^{-4} - z^{-5},$$

$$\begin{aligned} Y(z) &= X(z)\,H(z) \\ &= z^{-1} + 2z^{-2} - 2z^{-3} - 2z^{-4} + 2z^{-5} - 3z^{-6} + 2z^{-7}, \end{aligned}$$

$$\begin{aligned} y[n] &= \delta[n-1] + 2\delta[n-2] - 2\delta[n-3] - 2\delta[n-4] \\ &\quad + 2\delta[n-5] - 3\delta[n-6] + 2\delta[n-7]. \end{aligned}$$

Example 6.10 thus demonstrates that the convolution sum of an input $x[n]$ and the impulse response of a system $h[n]$ is indeed the output from the LTI system.

6.3.1 Impulse Response and BIBO Stability

We have discussed, in Chapter 5, the BIBO stability of continuous-time systems. The BIBO stability is also defined in discrete-time systems; for a *BIBO stable system*, bounded (finite) input signals always lead to bounded (finite) output signals. The BIBO stability of a system is expressed in terms of the impulse response of the system as

$$|y[n]| = |x[n] * h[n]| = \left| \sum_{k=-\infty}^{\infty} x[n-k]\,h[k] \right| \le \sum_{k=-\infty}^{\infty} |x[n-k]|\,|h[k]|.$$

Assuming that the input signal is bounded:

$$|x[n-k]| \le K,$$

with a positive contant K, the magnitude of the output should satisfy the following expression:

$$|y[k]| \le \sum_{k=-\infty}^{\infty} K\,|h[k]| \le K \sum_{k=-\infty}^{\infty} |h[k]|.$$

The above expression means that a discrete-time system is BIBO stable if its impulse response is absolutely summable such that

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{6.11}$$

6.3.2 Impulse Response and Causality

The concept of the impulse response enables one to describe the causality of a discrete-time system as

$$h[n] = 0 \quad (n < 0). \tag{6.12}$$

In other words, a *causal system* is one whose impulse response has no signal before the impulsive input at $n = 0$. For causal systems, calculating convolution can be a simpler task such that

$$\begin{aligned} y[n] = x[n] * h[n] &= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \\ &= \sum_{k=-\infty}^{n} x[k]\, h[n-k] + \sum_{k=n+1}^{\infty} x[k]\, h[n-k] \\ &= \sum_{k=-\infty}^{n} x[k]\, h[n-k]. \end{aligned}$$

Employing an extra assumption that $x[n] = 0$ for $n < 0$, the convolution sum can be expressed in a even simpler form as

$$y[n] = x[n] * h[n] = \left(\sum_{k=0}^{n} x[k]\, h[n-k] \right) u[n]. \tag{6.13}$$

Example 6.11 Determine the BIBO stability and causality of the systems that each the following impulse responses represent.

1. $h[n] = 0.5^n\, u[n-2]$
2. $h[n] = u[n]$
3. $h[n] = 2^n u[2-n]$
4. $h[n] = 2^n\, u[n+4]$

Solution

1. Stable / Causal

2. Unstable / Causal
3. Stable / Noncausal
4. Unstable / Noncausal

6.3.3 Impulse Response of Interconnected Systems

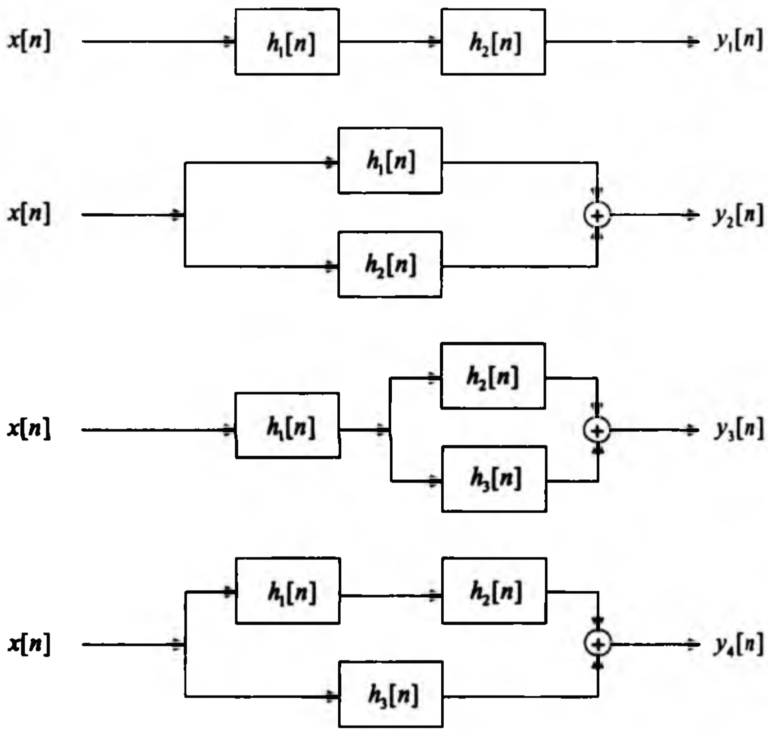


Figure 6.4: Interconnected convolution

Utilizing the concept of impulse response, we can describe characteristics of an *interconnected system*. Figure 6.4 shows a list of interconnected systems whose input-output relations are expressed as follows:

$$y_1[n] = x[n] * (h_1[n] * h_2[n]),$$

$$y_2[n] = x[n] * (h_1[n] + h_2[n]),$$

$$y_3[n] = x[n] * h_1[n] * (h_2[n] + h_3[n]),$$

$$y_4[n] = x[n] * (h_1[n] * h_2[n] + h_3[n]).$$

Example 6.12 Consider two subsystems that are described as

$$h_1[n] = 3\delta[n-1],$$
$$h_2[n] = 2\delta[n-2] + \delta[n-3].$$

Suppose one uses the following input sequence:

$$x[n] = \delta[n] - \delta[n-1],$$

and performs two experiments: the first with the series interconnection of the two subsystems, and the second with the parallel interconnection. What then are the output sequences from the two experiments? Denote the output from the first experiment as $y_1[n]$ and the output from the second as $y_2[n]$.

Solution

$$h_1[n] * h_2[n] = 3\delta[n-1] * (2\delta[n-2] + \delta[n-3])$$
$$= 6\delta[n-3] + 3\delta[n-4],$$

$$h_1[n] + h_2[n] = 3\delta[n-1] + 2\delta[n-2] + \delta[n-3].$$

$$y_1[n] = x[n] * (h_1[n] * h_2[n])$$
$$= (\delta[n] - \delta[n-1]) * (6\delta[n-3] + 3\delta[n-4]),$$
$$= 6\delta[n-3] - 3\delta[n-4] - 3\delta[n-5],$$

$$y_2[n] = x[n] * (h_1[n] + h_2[n])$$
$$= (\delta[n] - \delta[n-1]) * (3\delta[n-1] + 2\delta[n-2] + \delta[n-3]),$$
$$= 3\delta[n-1] - \delta[n-2] - \delta[n-3] - \delta[n-4].$$

6.4 NUMERICAL EXERCISE

We have dealt with the convolution sum of discrete-time sequences that are of infinite length but are analytically describable (Examples 6.1 - 6.5). We have also considered discrete-time sequences that are of finite length (Examples 6.6 - 6.8). The length of sequences has been limited so that one can manage algebraically. It is obvious that as the length of time sequences gets longer, we are quickly forced to use digital computers.

There are plenty of numerical packages that enable one to manage data sets of considerable size. Among them, MATLAB is the one that attract more and more scientists and engineers (Chaparro and Akan 2019; Karris 2008). We use MATLAB as the mean of introducing numerical works relevant to signal analysis. Those who are not familiar with MATLAB are recommended to review the basic usage of the package and essential grammar of MATLAB script (Hahn and Valentine 2019; Moore 2014).

6.4.1 Discrete-time Signals

The general expression of convolution sum is

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k].$$

In principle, it is impossible to calculate the above expression via a digital computer because of the infinite summation range of the convolution sum. We thus only consider signals that are of finite length and zero for $n < 0$, and we express the convolution sum as

$$y[n] = x[n] * h[n] = \left(\sum_{k=0}^{n} x[k]\, h[n-k] \right) u[n].$$

It is common to define time sequences via arrays in MATLAB. Consider the following script.

```
xn = [1 1 -2];
```

The script defines an array that corresponds to the following expression:

$$x[n] = \delta[n] + \delta[n-1] - 2\delta[n-2].$$

Note, however, that array indexing in MATLAB starts from 1 instead of 0. The first element of the array is thus referred to as xn(1) within MATLAB, while the first nonzero value of $x[n]$ occurs at $n = 0$. Consider also the following script.

```
hn = [0 1 1 -1 1 -1];
```

The script defines an array that corresponds to the following expression:

$$h[n] = \delta[n-1] + \delta[n-2] - \delta[n-3] + \delta[n-4] - \delta[n-5].$$

Note that the first nonzero value of $h[n]$ does not occur at $n = 0$, and, for the generation of an array that corresponds to $h[n]$, one has to manually assign the value of the first element of the array as 0.

Taking good care of indexing, one should find it is straightforward to calculate convolution and deconvolution via the following MATLAB commands:

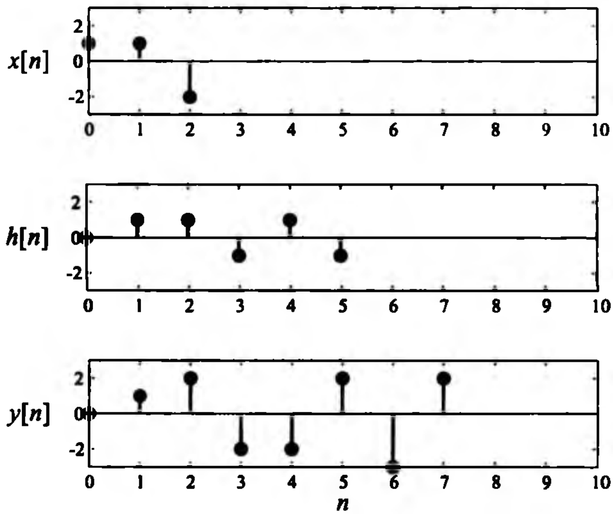
```
yn = conv(xn,hn);
hn = deconv(yn,xn);
```

Example 6.13 Type the following MATLAB script and run it. Check whether your run result satisfies the width property of discrete-time convolution (Figure 6.2).

```
clear;
xn = [1 1 -2];
hn = [0 1 1 -1 1 -1];
yn = conv(xn,hn);
nx = 0:length(xn)-1;
nh = 0:length(hn)-1;
ny = 0:length(yn)-1;

figure(1);
subplot(3,1,1);
stem(nx,xn,'Filled','Linewidth',1.5);
ylabel('x[n]');
```

```
set(get(gca,'ylabel'),'rotation',0);
axis([0 10 -3 3]);
subplot(3,1,2);
stem(nh,hn,'Filled','Linewidth',1.5);
ylabel('h[n]');
set(get(gca,'ylabel'),'rotation',0);
axis([0 10 -3 3]);
subplot(3,1,3);
stem(ny,yn,'Filled','Linewidth',1.5);
xlabel('n');
ylabel('y[n]');
set(get(gca,'ylabel'),'rotation',0);
axis([0 10 -3 3]);
```

Solution

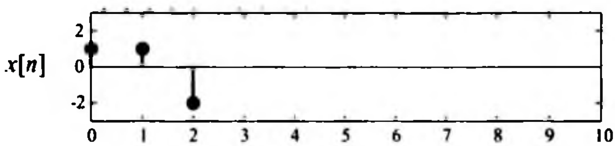


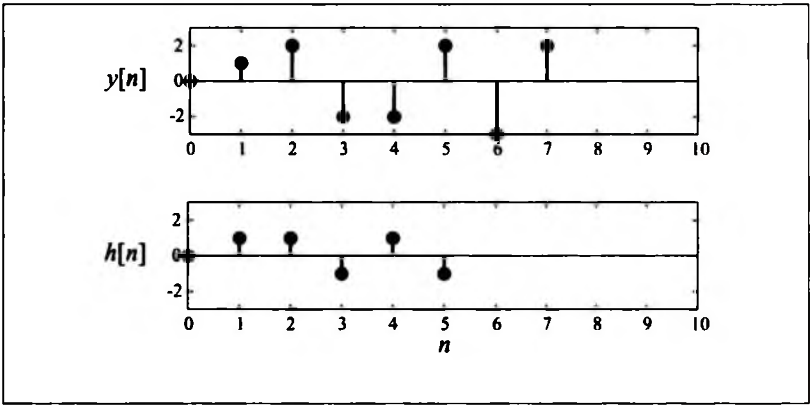
$N_x = 3$, $N_h = 5$, and $N_y = N_x + N_h - 1 = 7$. The width property is thus satisfied.

Example 6.14 Type the following MATLAB script and run it.

```
clear;
xn = [1 1 -2];
yn = [0 1 2 -2 -2 2 -3 2];
hn = deconv(yn,xn);
nx = 0:length(xn)-1;
ny = 0:length(yn)-1;
nh = 0:length(hn)-1;

figure(2);
subplot(3,1,1);
stem(nx,xn,'Filled','Linewidth',1.5);
ylabel('x[n]');
set(get(gca,'ylabel'),'rotation',0);
axis([0 10 -3 3]);
subplot(3,1,2);
stem(ny,yn,'Filled','Linewidth',1.5);
ylabel('y[n]');
set(get(gca,'ylabel'),'rotation',0);
axis([0 10 -3 3]);
subplot(3,1,3);
stem(nh,hn,'Filled','Linewidth',1.5);
xlabel('n');
ylabel('h[n]');
set(get(gca,'ylabel'),'rotation',0);
axis([0 10 -3 3]);
```

Solution





6.4.2 Continuous-time Signals

Reading the title that we are doing convolution integral via a digital computer, one might have misunderstood that we are trying to derive the analytic expression of a convolution integral result. No, we do not mean that. Instead of the analytic expression, we are trying to get a numerical data set that describes the convolution integral result.

The general expression of convolution integral is

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

In principle, it is impossible to calculate the above expression via a digital computer because of the integration range of the convolution integral. We thus only consider signals that are of finite length and zero for $t < 0$, and we express the convolution integral as

$$y(t) = x(t) * h(t) = \left[\int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau\right] u(t).$$

We then discretize $x(t)$ and $h(t)$ with a uniform discretization interval Δt and approximate the above integral as

$$y(n\Delta t) \approx \left[\Delta t \sum_{k=0}^{n\Delta t} x(k\Delta t)\, h(n\Delta t - k\Delta t)\right] u(n\Delta t).$$

The above expression can be, finally, transformed to an expression that only involves discrete-time sequences as follows:

$$y[n] = \Delta t \left(\sum_{k=0}^{n} x[k]\, h[n-k] \right) u[n] = \Delta t\, (x[n] * h[n]). \qquad (6.14)$$

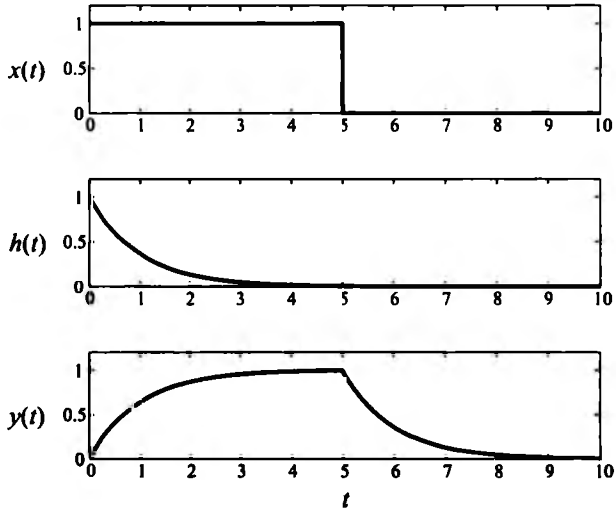
Expression 6.14 implies that to get a sequence that correctly describes the convolution integral result, one should multiply the discretization interval Δt to the convolution sum of two sequences, which, via the discretization, describe two continuous-time functions.

Example 6.15 Type the following MATLAB script and run it. Compare your run result with Figure 5.7.

```
clear;
dt = 0.01;                    % Discretization interval
tf = 10.0;                    % Maximum time range
ta = 0:dt:tf;                 % Time axis

xt = zeros(1,length(ta));
for id = 1:length(ta)
    if (ta(id) < 5.0)
        xt(id) = 1.0;
    end
end
ht = exp(-ta);
yc = dt*conv(xt,ht);         % Don't forget to MULTIPLY dt
yt = yc(1:length(ta));

figure(3);
subplot(3,1,1);
plot(ta,xt,'Linewidth',1.5);
axis([0 tf 0 1.2]);
ylabel('x(t)');
set(get(gca,'ylabel'),'rotation',0);
subplot(3,1,2);
plot(ta,ht,'Linewidth',1.5);
axis([0 tf 0 1.2]);
```

```
ylabel('h(t)');
set(get(gca,'ylabel'),'rotation',0);
subplot(3,1,3);
plot(ta,yt,'Linewidth',1.5);
axis([0 tf 0 1.2]);
xlabel('t');
ylabel('y(t)');
set(get(gca,'ylabel'),'rotation',0);
```

Solution



Example 6.15 demonstrates that continuous-time convolution can be approximated by discrete-time convolution. The expressions of the continuous-time signals in Example 6.15 are

$$x(t) = u(t)\,u(5-t),$$
$$h(t) = e^{-t}\,u(t),$$

and we are presenting their convolution integral result without any analytic calculation.

We have argued that for continuous-time signals, there is no direct mathematical way of defining deconvolution. But, with the help of digital computer, we are able to describe, at least numerically, the deconvolution of two continuous-time signals. Consider, for example, the following continuous-time signals:

$$x(t) = u(t)\,u(5-t),$$
$$y(t) = [1-e^{-t}]\,u(t)\,u(5-t) + e^{-t}\,[e^{5}-1]\,u(t-5).$$

There is no analytic method of deriving $h(t)$ that satisfies

$$y(t) = x(t) * h(t).$$

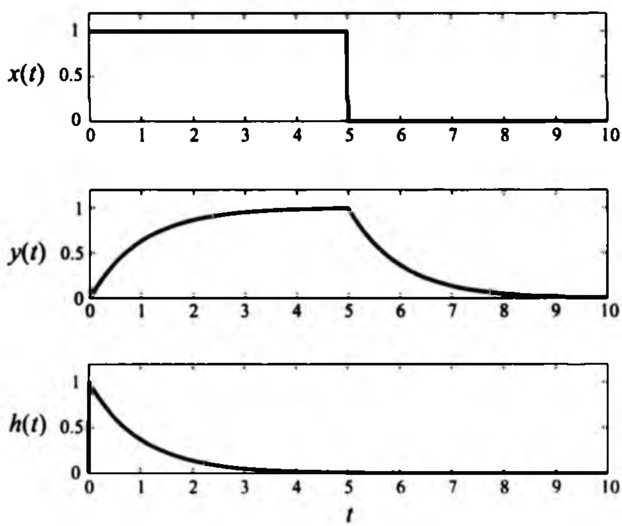
Example 6.16, however, demonstrates that one can numerically describe the continuous-time signal $h(t)$.

Example 6.16 Type the following MATLAB script and run it.

```
clear;
dt = 0.01;                % Discretization interval
tf = 10.0;                % Maximum time range
t1 = 0:dt:tf;             % Time axis for xt and ht
t2 = 0:dt:2*tf;           % Time axis for yt

xt = zeros(1,length(t1));
for id = 1:length(t1)
    if (t1(id) < 5.0)
        xt(id) = 1.0;
    end
end
yt = zeros(1,length(t2));
for id = 1:length(t2)
    if (t2(id) < 5.0)
        yt(id) = 1-exp(-t2(id));
    else
        yt(id) = exp(-t2(id))*(exp(5)-1);
    end
end
```

```
ht = deconv(yt,xt)/dt;  % Be careful to DIVIDE with dt

figure(4);
subplot(3,1,1);
plot(t1,xt,'Linewidth',1.5);
axis([0 tf 0 1.2]);
ylabel('x(t)');
set(get(gca,'ylabel'),'rotation',0);
subplot(3,1,2);
plot(t2,yt,'Linewidth',1.5);
axis([0 tf 0 1.2]);
ylabel('y(t)');
set(get(gca,'ylabel'),'rotation',0);
subplot(3,1,3);
plot(t1,ht,'Linewidth',1.5);
axis([0 tf 0 1.2]);
xlabel('t');
ylabel('h(t)');
set(get(gca,'ylabel'),'rotation',0);
```

Solution



PROBLEMS

Problem 6.1 Calculate the convolution of the following two sequences:

$$x[n] = u[n],$$
$$h[n] = n^2 u[n].$$

Problem 6.2 Calculate the convolution of the following two sequences:

$$x[n] = 3^n u[n],$$
$$h[n] = 2^n u[n].$$

Problem 6.3 Calculate the convolution of the following two sequences:

$$x[n] = a^n u[n],$$
$$h[n] = b^n u[n],$$

for $a \neq b$.

Problem 6.4 Calculate the convolution of the following two sequences:

$$x[n] = c^n u[n],$$
$$h[n] = c^n u[n].$$

Problem 6.5 Calculate the convolution of the following two sequences:

$$x[n] = \delta[n] - \delta[n-1],$$
$$h[n] = \delta[n] + 3\delta[n-1] + 2\delta[n-3].$$

Problem 6.6 Calculate the convolution of the following two sequences:

$$x[n] = \delta[n] - 3\delta[n-1] + 2\delta[n-3],$$
$$h[n] = \delta[n] + \delta[n-1].$$

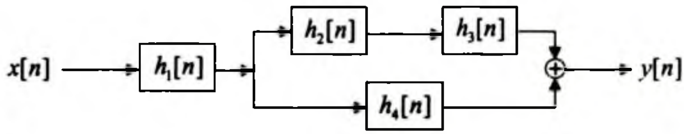
Problem 6.7 Find the impulse response $h[n]$ of a system that yields the following input / output sequences:

$$x[n] = \delta[n] + 2\delta[n-2],$$
$$y[n] = 2\delta[n] - 3\delta[n-1] + 4\delta[n-2] - 6\delta[n-3].$$

Problem 6.8 Find the impulse response $h[n]$ of a system that yields the following input / output sequences:

$$x[n] = \delta[n] - 2\delta[n-2],$$
$$y[n] = 2\delta[n-1] + \delta[n-2] - 5\delta[n-3] - \delta[n-4] + 2\delta[n-5] - 2\delta[n-6].$$

Problem 6.9 Consider an interconnected system shown below. Describe the relationship between $x[n]$ and $y[n]$ through convolution notation.



Problem 6.10 Sketch the block diagram that represents the following input-output relation:

$$y[n] = x[n] * (h_1[n] + h_2[n]) * h_3[n].$$

Chapter 7

CONCEPT OF FOURIER SERIES

In Chapters 5 and 6, we have discussed the time domain technique called convolution. For the rest of this study, we primarily focus on frequency domain techniques. The first topic in the frequency domain must be Fourier series, because it is the foundation on which we build the house of Fourier analysis. Noticing lost in the middle of discussing Fourier analysis, readers need to come back to Fourier series and ponder again its significance. Those who are not yet familiar with handling complex numbers are also advised to review the basic theory of complex numbers (Appendix C) ahead of discussing the Fourier series.

7.1 INTRODUCTION TO FOURIER SERIES

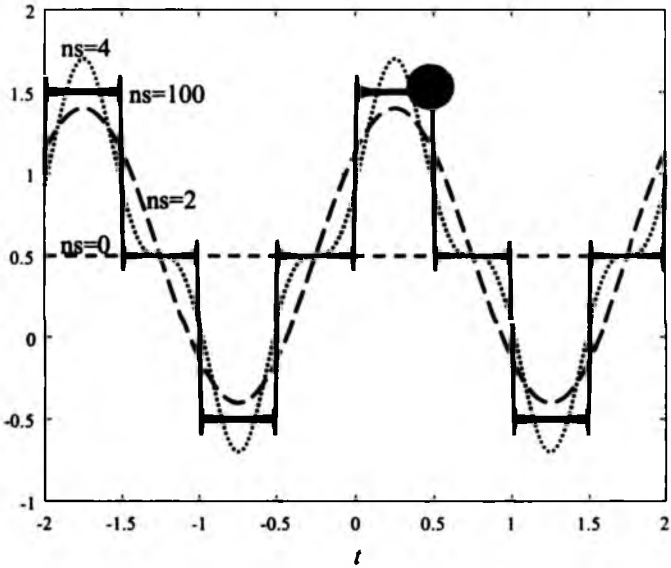
A good way of substantiating an unfamiliar concept is to experience something tangible about the concept. We thus first introduce an example and then discuss what Fourier series is all about. Readers are strongly recommended to do the example ahead of anything else.

Example 7.1 Type the following MATLAB script and run it. Try different values of `ns` (for example, `ns = 0, 2, 4, 10, 100, 1000`) and observe their results.

```
clear;
ns = 100;
dt = 0.01;
tf = 2.00;
ta = -tf:dt:tf;
nt = length(ta);

fs = 0.5*ones(1,nt);
for id = 1:ns
    if mod(id,4) == 1
```

```
        fs = fs + 2.0*cos(id*pi*ta)/(id*pi);
        fs = fs + 2.0*sin(id*pi*ta)/(id*pi);
    elseif mod(id,4) == 3
        fs = fs - 2.0*cos(id*pi*ta)/(id*pi);
        fs = fs + 2.0*sin(id*pi*ta)/(id*pi);
    end
end

figure(1);
plot(ta,fs,'Linewidth',1.5);
xlabel('t');
axis([-tf tf -1 2]);
```

Solution



Example 7.1 shows that one can add a lot of trigonometric functions to build up a periodic function shown in Figure 7.1. It also shows that near the discontinuous jumps of Figure 7.1, graphs in Example 7.1 may exhibit overshot (gray background), which is called the *Gibbs phenomenon*.

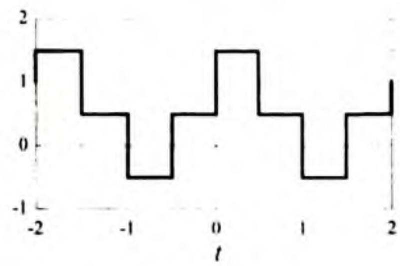


Figure 7.1: A periodic function

While keeping the `dt` value as 0.01, one can remove the overshot with a larger value of `ns`, such as 1000. The type of periodic functions that trigonometric functions may synthesize are not limited to the one shown in Figure 7.1. To the contrary, trigonometric functions can synthesize a variety of different periodic functions. In other words, a lot of periodic functions can be expressed in terms of trigonometric functions. This process of decomposing a *periodic signal* into a series of trigonometric functions is known as the *Fourier series* expansion. The Fourier series expansion is not possible for every periodic function. A periodic function $x(t)$ can be expanded as a Fourier series only if it fulfills the *Dirichlet conditions* summarized below (Oppenheim and Willsky, 1997).

1. $x(t)$ must be absolutely integrable over a period.

2. $x(t)$ must be of bounded variation in any given bounded interval.

3. $x(t)$ must have a finite number of discontinuities in any given bounded interval, and the discontinuities cannot be infinite.

7.2 TRIGONOMETRIC FORM OF FOURIER SERIES

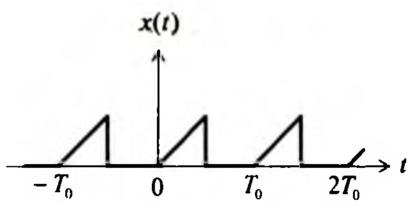


Figure 7.2: Periodic function and fundamental period

Figure 7.2 shows an example of periodic function that can be expanded as a Fourier series. The *fundamental period*, which is the smallest possible period value, is denoted as T_0, and the *fundamental frequency* Ω is expressed as

$$\Omega = \frac{2\pi}{T_0}. \tag{7.1}$$

While expanding a periodic function as a Fourier series, we only consider trigonometric functions whose angular frequencies are expressed as

$$\omega_m = m\Omega, \tag{7.2}$$

where m is a natural number. In other words, a periodic function $x(t)$ with the fundamental frequency Ω can be expanded as

$$\begin{aligned} x(t) = a_0 &+ a_1 \cos(\Omega t) + a_2 \cos(2\Omega t) + a_3 \cos(3\Omega t) + \cdots \\ &+ b_1 \sin(\Omega t) + b_2 \sin(2\Omega t) + b_3 \cos(3\Omega t) + \cdots . \end{aligned}$$

or, in short,

$$x(t) = a_0 + \sum_{m=1}^{\infty} [a_m \cos(m\Omega t) + b_m \sin(m\Omega t)]. \tag{7.3}$$

Expression 7.3 is known as the *trigonometric form* of Fourier series. Note that the coefficient a_0 represents zero frequency or DC component of the periodic function. Note also that cosine functions build up the even part of $x(t)$ and sine functions synthesize the odd part of the periodic function.

Having set up the mathematical prototype of Fourier series, the remaining issue is how to determine the coefficients a_m and b_m. We first consider a case that a periodic function can readily be expressed in terms of trigonometric functions, and then study the general approach of determining coefficients of the Fourier series.

Example 7.2 Determine the fundamental frequency and Fourier series coefficients of the following periodic function:

$$x(t) = \sin(\pi t) + \sin^2(\pi t).$$

Solution

$$x(t) = \sin(\pi t) + \frac{1 - \cos(2\pi t)}{2}.$$

It is obvious $\Omega = \pi$, and we express

$$x(t) = \frac{1}{2} - \frac{1}{2}\cos(2\Omega t) + \sin(\Omega t).$$

Three nonzero coefficients of the Fourier series are thus

$$a_0 = \frac{1}{2}, \quad a_2 = -\frac{1}{2}, \quad \text{and} \quad b_1 = 1.$$

Example 7.3 Determine the fundamental frequency and Fourier series coefficients of the following Fourier series expression:

$$x(t) = \sum_{m=1}^{\infty} \frac{4}{m} \sin^2(m\pi/2)\cos(4m\pi t).$$

Solution

$$x(t) = 4\sin^2\frac{\pi}{2}\cos(4\pi t) + 2\sin^2\pi\cos(8\pi t) + \frac{4}{3}\sin^2\frac{3\pi}{2}\cos(12\pi t) + \sin^2 2\pi\cos(16\pi t) + \cdots.$$

It is obvious $\Omega = 4\pi$, and we express

$$x(t) = 4\sin^2\frac{\pi}{2}\cos(\Omega t) + 2\sin^2\pi\cos(2\Omega t) + \frac{4}{3}\sin^2\frac{3\pi}{2}\cos(3\Omega t) + \sin^2 2\pi\cos(4\Omega t) + \cdots.$$

The Fourier series coefficients are thus

$$a_0 = 0, \quad a_m = \frac{4}{m}\sin^2\frac{m\pi}{2}, \quad \text{and} \quad b_m = 0.$$

7.2.1 Orthogonality of Trigonometric Functions

The general approach of determining the Fourier series coefficients is a consequence of the concept called the *trigonometric orthogonality*. We thus briefly discuss the concept first and then introduce the approach of determining coefficient of the Fourier series.

In mathematics, *orthogonality* is the generalization of the notion of perpendicularity to the linear algebra of bilinear forms (Friedberg *et al.* 2002). And the orthogonality is generally represented as zero inner product. In a function space, inner product of two functions f and g can be described as

$$< f, g > = \int_a^b f(t)\, g(t)\, dt,$$

and the two functions are called *orthogonal functions* if the inner product equals zero (Kreyszig 2011; Snieder and van Wijk 2015).

The orthogonality of trigonometric functions are the foundation of the Fourier analysis, and we summarize the concept as follows:

$$\int_{t_0}^{t_0+T_0} \cos(m\Omega t)\, dt = 0, \tag{7.4}$$

$$\int_{t_0}^{t_0+T_0} \sin(m\Omega t)\, dt = 0, \tag{7.5}$$

$$\int_{t_0}^{t_0+T_0} \cos(m\Omega t)\, \cos(n\Omega t)\, dt = \begin{cases} 0 & (m \neq n), \\ T_0/2 & (m = n), \end{cases} \tag{7.6}$$

$$\int_{t_0}^{t_0+T_0} \sin(m\Omega t)\, \sin(n\Omega t)\, dt = \begin{cases} 0 & (m \neq n), \\ T_0/2 & (m = n), \end{cases} \tag{7.7}$$

$$\int_{t_0}^{t_0+T_0} \cos(m\Omega t)\, \sin(n\Omega t)\, dt = 0, \tag{7.8}$$

where m and n are natural numbers. Note that the integration may begin at an arbitrary time t_0 but should last for a complete cycle of the fundamental period T_0. We do not provide mathematical proof of the above expressions. Instead of that, we give an intuitive explanation as shown in Figure 7.3.

Figure 7.3 (a) demonstrates that with $t_0 = 0$,

$$\int_{t_0}^{t_0+T_0} \sin(5\Omega t)\, dt = 0.$$

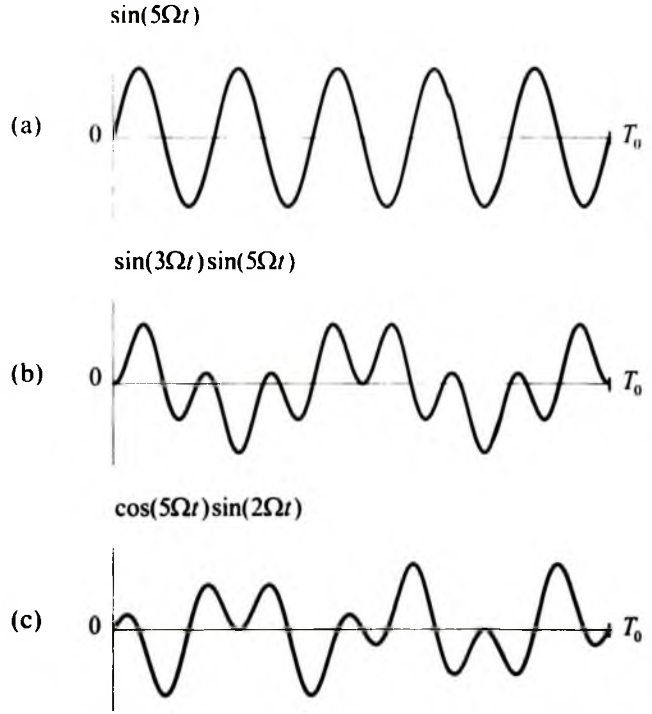


Figure 7.3: A demonstration of the orthogonality of trigonometric functions

We also note that as m increases, the period of $\cos(m\Omega t)$ and $\sin(m\Omega t)$ decreases as

$$T_m = \frac{2\pi}{\omega_m} = \frac{2\pi}{m\Omega} = \frac{T_0}{m}. \tag{7.9}$$

In other words, $\cos(m\Omega t)$ and $\sin(m\Omega t)$ repeat exactly integer number of cycles for $t_0 < t < t_0 + T_0$, and their integration results over a fundamental period must be zero. We are thus convinced that expressions 7.4 and 7.5 are correct.

Figure 7.3 (b) suggests that with $t_0 = 0$,

$$\int_{t_0}^{t_0+T_0} \sin(3\Omega t)\ \sin(5\Omega t)\, dt = 0,$$

and, with a similar argument to the above case, we are convinced that expressions 7.6 - 7.7 are correct for $m \neq n$. For $m = n$, expression 7.6 can be proved as follows:

$$\begin{aligned}
\int_{t_0}^{t_0+T_0} \cos^2(m\Omega t)\, dt &= \int_{t_0}^{t_0+T_0} \frac{1+\cos(2m\Omega t)}{2}\, dt \\
&= \left[\frac{t}{2}\right]_{t_0}^{t_0+T_0} + \left|\frac{\sin(2m\Omega t))}{4m\Omega}\right|_{t_0}^{t_0+T_0} \\
&= \frac{T_0}{2} + \frac{1}{4m\Omega}[\sin(2m\Omega t_0 + 2m\Omega T_0) - \sin(2m\Omega t_0)] \\
&= \frac{T_0}{2} + \frac{1}{4m\Omega}[\sin(2m\Omega t_0 + 4\pi m) - \sin(2m\Omega t_0)] \\
&= \frac{T_0}{2} + \frac{1}{4m\Omega}[\sin(2m\Omega t_0) - \sin(2m\Omega t_0)] = \frac{T_0}{2}.
\end{aligned}$$

And it is evident that for $m = n$, expression 7.7 is also correct.

Finally, Figure 7.3 (c) demonstrates that with $t_0 = 0$,

$$\int_{t_0}^{t_0+T_0} \cos(5\Omega t)\ \sin(2\Omega t)\, dt = 0,$$

and convinces one that expression 7.8 is correct.

7.2.2 Coefficients of Fourier Series

With the trigonometric orthogonality in expressions 7.4 - 7.8 ready for use, it is straightforward to determine the Fourier series coefficient in expression 7.3.

We determine a_0 by integrating expression 7.3 over a fundamental period as follows:

$$\begin{aligned}
\int_{t_0}^{t_0+T_0} x(t)\, dt &= a_0 \int_{t_0}^{t_0+T_0} dt \\
&\quad + \sum_{m=1}^{\infty} \left[a_m \int_{t_0}^{t_0+T_0} \cos(m\Omega t)\, dt + b_m \int_{t_0}^{t_0+T_0} \sin(m\Omega t)\, dt\right].
\end{aligned}$$

Expressions 7.4 and 7.5 simplify the above expression as

$$\int_{t_0}^{t_0+T_0} x(t)\, dt = a_0 T_0,$$

and we express the coefficient a_0 as

$$a_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)\, dt. \tag{7.10}$$

For the determination of the coefficient a_m, we first multiply $\cos(n\Omega t)$ to expression 7.3 and integrate over a fundamental period as follows:

$$\int_{t_0}^{t_0+T_0} x(t)\, \cos(n\Omega t)\, dt = a_0 \int_{t_0}^{t_0+T_0} \cos(n\Omega t)\, dt$$
$$+\sum_{m=1}^{\infty} \left[a_m \int_{t_0}^{t_0+T_0} \cos(m\Omega t) \cos(n\Omega t)\, dt + b_m \int_{t_0}^{t_0+T_0} \sin(m\Omega t) \cos(n\Omega t)\, dt \right].$$

Trigonometric orthogonality now comes into play and enables one to write as

$$\int_{t_0}^{t_0+T_0} x(t)\, \cos(n\Omega t)\, dt = \sum_{m=1}^{\infty} a_m \int_{t_0}^{t_0+T_0} \cos(m\Omega t)\, \cos(n\Omega t)\, dt$$
$$= a_n \int_{t_0}^{t_0+T_0} \cos^2(n\Omega t)\, dt = \frac{a_n T_0}{2}.$$

We thus express a_m as

$$a_m = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t)\, \cos(m\Omega t)\, dt. \tag{7.11}$$

Similarly, multiplying $\sin(n\Omega t)$ to expression 7.3 and integrating over a fundamental period yields the following expression:

$$\int_{t_0}^{t_0+T_0} x(t)\, \sin(n\Omega t)\, dt = a_0 \int_{t_0}^{t_0+T_0} \sin(n\Omega t)\, dt$$
$$+\sum_{m=1}^{\infty} \left[a_m \int_{t_0}^{t_0+T_0} \cos(m\Omega t) \sin(n\Omega t)\, dt + b_m \int_{t_0}^{t_0+T_0} \sin(m\Omega t) \sin(n\Omega t)\, dt \right].$$

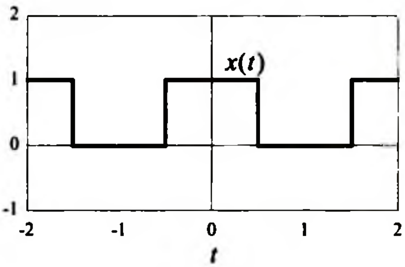
We then simplify the above expression as

$$\int_{t_0}^{t_0+T_0} x(t)\, \sin(n\Omega t)\, dt = \sum_{m=1}^{\infty} b_m \int_{t_0}^{t_0+T_0} \sin(m\Omega t)\, \sin(n\Omega t)\, dt$$
$$= b_n \int_{t_0}^{t_0+T_0} \sin^2(n\Omega t)\, dt = \frac{b_n T_0}{2},$$

and we express b_m as

$$b_m = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,\sin(m\Omega t)\,dt. \tag{7.12}$$

Example 7.4 Determine the fundamental frequency and Fourier series coefficients of the periodic function shown below.



Solution

The fundamental period T_0 is 2, and the fundamental frequency is given as $\Omega = 2\pi/T_0 = \pi$. For the derivation of Fourier series coefficient, we take the integration range between 0 and T_0.

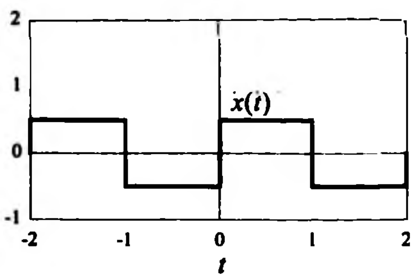
$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{2}\int_0^{2} x(t)\,dt = \frac{1}{2}.$$

$$\begin{aligned}
a_m &= \frac{2}{T_0}\int_0^{T_0} x(t)\,\cos(m\Omega t)\,dt = \int_0^{2} x(t)\,\cos(m\pi t)\,dt \\
&= \int_0^{1/2} \cos(m\pi t)\,dt + \int_{3/2}^{2} \cos(m\pi t)\,dt \\
&= \left[\frac{\sin(m\pi t)}{m\pi}\right]_0^{1/2} + \left[\frac{\sin(m\pi t)}{m\pi}\right]_{3/2}^{2} \\
&= \frac{\sin(m\pi/2) - \sin(0) + \sin(2m\pi) - \sin(3m\pi/2)}{m\pi} \\
&= \frac{\sin(m\pi/2) - \sin(3m\pi/2)}{m\pi} = \frac{2\sin(m\pi/2)}{m\pi}
\end{aligned}$$

$$= \begin{cases} 2/(m\pi) & (m = 1, 5, 9, \cdots), \\ 0 & (m = 2, 4, 6, \cdots), \\ -2/(m\pi) & (m = 3, 7, 11, \cdots). \end{cases}$$

$$\begin{aligned} b_m &= \frac{2}{T_0} \int_0^{T_0} x(t) \sin(m\Omega t)\, dt = \int_0^2 x(t) \sin(m\pi t)\, dt \\ &= \int_0^{1/2} \sin(m\pi t)\, dt + \int_{3/2}^2 \sin(m\pi t)\, dt \\ &= \left[-\frac{\cos(m\pi t)}{m\pi} \right]_0^{1/2} + \left[-\frac{\cos(m\pi t)}{m\pi} \right]_{3/2}^2 \\ &= \frac{\cos(0) - \cos(m\pi/2) + \cos(3m\pi/2) - \cos(2m\pi)}{m\pi} \\ &= 0. \end{aligned}$$

Example 7.5 Determine the fundamental frequency and Fourier series coefficients of the periodic function shown below.



Solution

The fundamental period T_0 is 2, and the fundamental frequency is given as $\Omega = 2\pi/T_0 = \pi$. For the derivation of Fourier series coefficient, we take the integration range between 0 and T_0.

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt = \frac{1}{2} \int_0^{2} x(t)\, dt = 0.$$

$$\begin{aligned}
a_m &= \frac{2}{T_0} \int_0^{T_0} x(t)\ \cos(m\Omega t)\, dt = \int_0^{2} x(t)\ \cos(m\pi t)\, dt \\
&= \int_0^{1} \frac{\cos(m\pi t)}{2}\, dt - \int_1^{2} \frac{\cos(m\pi t)}{2}\, dt \\
&= \left[\frac{\sin(m\pi t)}{2m\pi}\right]_0^1 - \left[\frac{\sin(m\pi t)}{2m\pi}\right]_1^2 \\
&= \frac{\sin(m\pi) - \sin(0) + \sin(m\pi) - \sin(2m\pi)}{2m\pi} = 0 .
\end{aligned}$$

$$\begin{aligned}
b_m &= \frac{2}{T_0} \int_0^{T_0} x(t)\ \sin(m\Omega t)\, dt = \int_0^{2} x(t)\ \sin(m\pi t)\, dt \\
&= \int_0^{1} \frac{\sin(m\pi t)}{2}\, dt - \int_1^{2} \frac{\sin(m\pi t)}{2}\, dt \\
&= \left[-\frac{\cos(m\pi t)}{2m\pi}\right]_0^1 + \left[\frac{\cos(m\pi t)}{2m\pi}\right]_1^2 \\
&= \frac{\cos(0) - \cos(m\pi) + \cos(2m\pi) - \cos(m\pi)}{2m\pi} \\
&= \frac{1 - \cos(m\pi)}{m\pi} = \begin{cases} 2/(m\pi) & (m = 1, 3, 5, \cdots), \\ 0 & (m = 2, 4, 6, \cdots). \end{cases}
\end{aligned}$$

Examples 7.4 and 7.5 demonstrate that the Fourier series expansion of an even function does not require sine functions ($b_m = 0$), while the expansion of an odd function does not require cosine functions ($a_m = 0$). Generally speaking, while synthesizing a periodic function via Fourier series, cosine functions contribute on the even part of the function whereas sine functions build up the odd part of the function. Incidentally, the two models in Examples 7.4 and 7.5 represent the even and odd parts of the model in Figure 7.1.

7.3 AMPLITUDE-PHASE FORM OF FOURIER SERIES

We have so far expressed Fourier series in the trigonometric form, which, in general, uses both cosine and sine functions at the same time to represent a frequency component. In fact, we can reduce the number of functions and use only one function for each frequency component as follows:

$$x(t) = \sum_{m=0}^{\infty} A_m \cos(m\Omega t + \varphi_m). \tag{7.13}$$

Expression 7.13 is called the *amplitude-phase form* of Fourier series. To associate the two different forms with each other, we rewrite expression 7.13 as

$$\begin{aligned} x(t) &= A_0 \cos(\varphi_0) + A_1 \cos(\Omega t + \varphi_1) + A_2 \cos(2\Omega t + \varphi_2) + \cdots \\ &= A_0 \cos(\varphi_0) + A_1 \cos(\Omega t) \cos(\varphi_1) + A_2 \cos(2\Omega t) \cos(\varphi_2) + \cdots \\ &\qquad - A_1 \sin(\Omega t) \sin(\varphi_1) - A_2 \sin(2\Omega t) \sin(\varphi_2) - \cdots . \end{aligned}$$

Comparing the above expression with expression 7.3, we make connection between the two different types of coefficients as

$$a_0 = A_0 \cos\varphi_0, \quad a_m = A_m \cos\varphi_m, \quad b_m = -A_m \sin\varphi_m,$$

or, equivalently as,

$$A_0 = |a_0|, \tag{7.14}$$

$$\varphi_0 = \begin{cases} 0 & (a_0 \geq 0), \\ \pm\pi & (a_0 < 0), \end{cases} \tag{7.15}$$

$$A_m = \sqrt{a_m^2 + b_m^2}, \tag{7.16}$$

$$\varphi_m = -\tan^{-1}(b_m/a_m). \tag{7.17}$$

The amplitude-phase form is a powerful way of representing Fourier series and suggests one to ponder again the meaning of Fourier series. We have argued that the Fourier series expansion is to decompose any periodic function into a combination of sinusoidal functions. And each different sinusoidal functions represent different frequency components with their own amplitude and phase values. In other words, the two coefficients A_m and φ_m are the ones that exhibit the *amplitude spectrum* and *phase spectrum* of the original periodic function.

Example 7.6 Sketch the amplitude and phase spectra of the following periodic function:

$$x(t) = -2 + 4\cos(8\pi t + \pi/6) + \sin(12\pi t) - 3\sin(20\pi t).$$

Solution

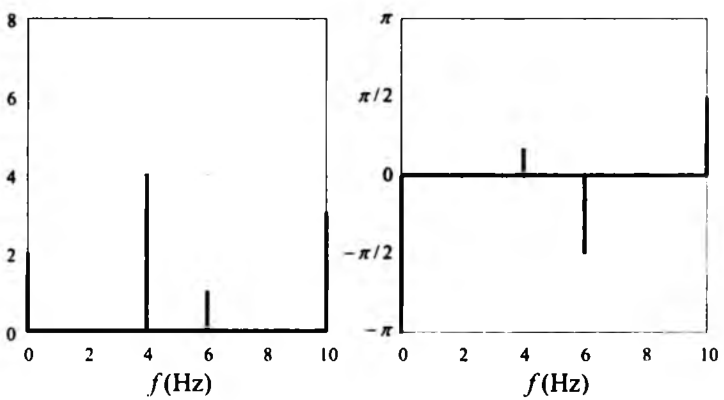
With $\Omega = 4\pi$, we write $x(t)$ as

$$x(t) = 2\cos(-\pi) + 4\cos(2\Omega t + \pi/6) + \cos(3\Omega t - \pi/2) + 3\cos(5\Omega t + \pi/2),$$

and identify the coefficients A_m and φ_m as

$$A_0 = 2, \quad A_2 = 4, \quad A_3 = 1, \quad A_5 = 3,$$
$$\varphi_0 = -\pi, \quad \varphi_2 = \pi/6, \quad \varphi_3 = -\pi/2, \quad \varphi_5 = \pi/2.$$

We finally determine $T_0 = 2\pi/\Omega = 1/2$, take $\Delta f = 1/T_0 = 2$, and sketch the amplitude and phase spectra as follows:



Example 7.6 well demonstrates the concept of amplitude and phase spectra, but the original periodic function $x(t)$ is already expressed as a combination of sinusoidal functions. For a more general example, consider the periodic function in Figure 7.1. We do not need to derive the Fourier series coefficients of the model, because the two models in Examples 7.4 and 7.5 are the even and odd parts of the model in Figure 7.1. The Fourier series coefficients of the model in Figure 7.1 is derived by simply adding the results of Examples 7.4 and 7.5 such that

$$a_0 = 1/2,$$

$$a_m = \begin{cases} 2/(m\pi) & (m = 1, 5, 9, \cdots), \\ 0 & (m = 2, 4, 6, \cdots), \\ -2/(m\pi) & (m = 3, 7, 11, \cdots), \end{cases}$$

$$b_m = \begin{cases} 2/(m\pi) & (m = 1, 5, 9, \cdots), \\ 0 & (m = 2, 4, 6, \cdots), \\ 2/(m\pi) & (m = 3, 7, 11, \cdots). \end{cases}$$

The coefficients of the amplitude-phase form are thus expressed as

$$A_0 = 1/2,$$

$$\varphi_0 = 0,$$

$$A_m = \begin{cases} 2\sqrt{2}/(m\pi) & (m = 1, 5, 9, \cdots), \\ 0 & (m = 2, 4, 6, \cdots), \\ 2\sqrt{2}/(m\pi) & (m = 3, 7, 11, \cdots), \end{cases}$$

$$\varphi_m = \begin{cases} -\pi/4 & (m = 1, 5, 9, \cdots), \\ 0 & (m = 2, 4, 6, \cdots), \\ -3\pi/4 & (m = 3, 7, 11, \cdots). \end{cases}$$

Figure 7.4 shows the amplitude and phase spectra of the model shown in Figure 7.1. Note that the horizontal axes have been converted from the series index m to the linear frequency f. The relation between the series index and the frequency is expressed as

$$f_m = m\,\Delta f = \frac{m}{T_0}. \tag{7.18}$$

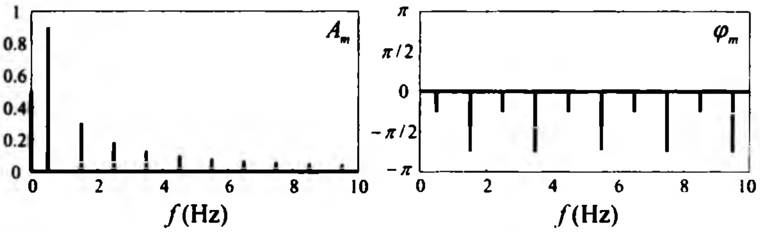


Figure 7.4: Amplitude and phase spectra of the model shown in Figure 7.1. The fundamental period T_0 of the model is 2, and thus the frequency interval Δf is 0.5.

7.4 EXPONENTIAL FORM OF FOURIER SERIES

Two different forms of Fourier series have been introduced. There is, in fact, one more form that is called the *exponential form* of Fourier series. And representing Fourier series in the exponential form is the foundation of the remaining discussion about Fourier analysis.

Recalling the *Euler's formula* in Appendix C and expressions C.7 - C.8, one may rewrite the trigonometric functions in expression 7.3 as

$$\cos(m\Omega t) = \frac{e^{jm\Omega t} + e^{-jm\Omega t}}{2},$$
$$\sin(m\Omega t) = \frac{e^{jm\Omega t} - e^{-jm\Omega t}}{2j} = \frac{je^{-jm\Omega t} - je^{jm\Omega t}}{2}.$$

The trigonometric form of Fourier series then transforms to

$$x(t) = a_0 + \sum_{m=1}^{\infty}\left[a_m\frac{e^{jm\Omega t} + e^{-jm\Omega t}}{2} + b_m\frac{je^{-jm\Omega t} - je^{jm\Omega t}}{2}\right]$$
$$= a_0 + \sum_{m=1}^{\infty}\left[\frac{a_m - jb_m}{2}e^{jm\Omega t}\right] + \sum_{m=1}^{\infty}\left[\frac{a_m + jb_m}{2}e^{-jm\Omega t}\right].$$

Denoting

$$X[0] = a_0, \quad X[m] = \frac{a_m - jb_m}{2}, \quad \text{and} \quad X[-m] = \frac{a_m + jb_m}{2}, \tag{7.19}$$

respectively, we first rewrite the expression of Fourier series as

$$x(t) = X[0] + \sum_{m=1}^{\infty} X[m]\, e^{jm\Omega t} + \sum_{m=1}^{\infty} X[-m]\, e^{-jm\Omega t}$$

and further simplify the expression as

$$x(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t}. \qquad (7.20)$$

Expression 7.20 is the exponential form of Fourier series. The significance of the sequence $X[m]$ is not clarified yet. To better substantiate the significance of the sequence $X[m]$, we recall expressions 7.10 - 7.12 and evaluate the three quantities $X[0]$, $X[m]$, and $X[-m]$ as follows:

$$X[0] = a_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\, e^{j0\Omega t}\,dt,$$

$$\begin{aligned} X[m] &= \frac{a_m - jb_m}{2} \\ &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,\cos(m\Omega t)\,dt - \frac{j}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,\sin(m\Omega t)\,dt \\ &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,[\cos(m\Omega t) - j\sin(m\Omega t)]\,dt \\ &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\, e^{-jm\Omega t}\,dt, \end{aligned}$$

and

$$\begin{aligned} X[-m] &= \frac{a_m + jb_m}{2} \\ &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,\cos(m\Omega t)\,dt + \frac{j}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,\sin(m\Omega t)\,dt \\ &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,[\cos(m\Omega t) + j\sin(m\Omega t)]\,dt \\ &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\, e^{jm\Omega t}\,dt. \end{aligned}$$

It is obvious from the above evaluations that regardless of the value of m, the sequence $X[m]$ can be always expressed as

$$X[m] = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\, e^{-jm\Omega t}\,dt. \qquad (7.21)$$

Example 7.7 Obtain the exponential Fourier series coefficients of the following function:

$$x(t) = 1 + \cos^3(2\pi t).$$

Solution

The fundamental frequency $\Omega = 2\pi$.

$$x(t) = 1 + \left[\frac{e^{j\Omega t} + e^{-j\Omega t}}{2}\right]^3$$

$$= 1 + \frac{1}{8}[e^{j3\Omega t} + 3e^{j\Omega t} + 3e^{-j\Omega t} + e^{-j3\Omega t}]$$

$$= \frac{1}{8}e^{-j3\Omega t} + \frac{3}{8}e^{-j\Omega t} + 1 + \frac{3}{8}e^{j\Omega t} + \frac{1}{8}e^{j3\Omega t}.$$

Therefore, $X[-3] = 1/8$, $X[-1] = 3/8$, $X[0] = 1$, $X[1] = 3/8$, and $X[3] = 1/8$.

Example 7.8 Determine the Fourier series coefficients $X[m]$ of the model shown in Figure 7.1.

Solution

With $T_0 = 2$, $\Omega = \pi$, we take the integration range between 0 and T_0.

$$X[m] = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jm\Omega t}\, dt = \frac{1}{2}\int_0^{2} x(t)\, e^{-jm\pi t}\, dt$$

$$= \frac{3}{4}\int_0^{1/2} e^{-jm\pi t}\, dt + \frac{1}{4}\int_{1/2}^{1} e^{-jm\pi t}\, dt$$

$$- \frac{1}{4}\int_1^{3/2} e^{-jm\pi t}\, dt + \frac{1}{4}\int_{3/2}^{2} e^{-jm\pi t}\, dt$$

$$= -\frac{3}{4}\left[\frac{e^{-jm\pi t}}{jm\pi}\right]_0^{1/2} - \frac{1}{4}\left[\frac{e^{-jm\pi t}}{jm\pi}\right]_{1/2}^{1}$$

$$+ \frac{1}{4}\left[\frac{e^{-jm\pi t}}{jm\pi}\right]_1^{3/2} - \frac{1}{4}\left[\frac{e^{-jm\pi t}}{jm\pi}\right]_{3/2}^{2}$$

$$= \frac{3j[e^{-jm\pi/2} - 1]}{4m\pi} + \frac{j[e^{-jm\pi} - e^{-jm\pi/2}]}{4m\pi}$$

$$- \frac{j[e^{-j3m\pi/2} - e^{-jm\pi}]}{4m\pi} + \frac{j[e^{-j2m\pi} - e^{-j3m\pi/2}]}{4m\pi}$$

$$= \frac{3j[(-j)^m - 1]}{4m\pi} + \frac{j[(-1)^m - (-j)^m]}{4m\pi}$$

$$- \frac{j[(j)^m - (-1)^m]}{4m\pi} + \frac{j[1 - (j)^m]}{4m\pi}$$

$$= \frac{j[-1 + (-j)^m + (-1)^m - (j)^m]}{2m\pi}.$$

The above expression is, however, not suitable for $m = 0$. We thus separately evaluate $X[0]$ as follows:

$$X[0] = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{2}\int_0^{2} x(t)\,dt$$

$$= \frac{3}{4}\int_0^{1/2} dt + \frac{1}{4}\int_{1/2}^{1} dt - \frac{1}{4}\int_1^{3/2} dt + \frac{1}{4}\int_{3/2}^{2} dt = \frac{1}{2}.$$

7.4.1 Amplitude and Phase Spectra

We have argued that with the amplitude-phase form, the amplitude and phase spectra of a periodic function is represented by A_m and φ_m. With the exponential form of Fourier series, the amplitude and phase information of the periodic function $x(t)$ is encapsulated in the complex quantity $X[m]$. Expressing the sequence $X[m]$ as

$$X[m] = |X[m]|\, e^{j\theta_m}, \tag{7.22}$$

we identify that $|X[m]|$ and θ_m are the ones that represent the *amplitude spectrum* and *phase spectrum* of $x(t)$.

In Example 7.8, we have studied that the Fourier series coefficient $X[m]$ of the model in Figure 7.1 is expressed as

$$X[m] = \frac{j[-1 + (-j)^m + (-1)^m - (j)^m]}{2m\pi}.$$

Taking the amplitude $|X[m]|$ and phase θ_m of the complex quantity, we can display the amplitude and phase spectra of the periodic function as shown in Figure 7.5. Note that we also display the amplitude A_m and phase φ_m for comparison. It is well demonstrated in Figure 7.5 that amplitude and phase spectra of the exponential form are, in principle, equivalent to those of the amplitude-phase form. Main difference between the two is that in the case of the exponential form, the frequency axis is extended to the negative values. And, while extending the frequency axis of display, amplitude values are reduced to the half and copy-pasted with even symmetry around the zero frequency. Phase values are, on the other hand, not altered and copy-pasted with odd symmetry. In other words, amplitude spectra $|X[m]|$ should always maintain even symmetry, whereas phase spectra θ_m should always exhibit odd symmetry. Incidentally, the physical significance of negative frequency is the opposite direction of rotation or wave movement.

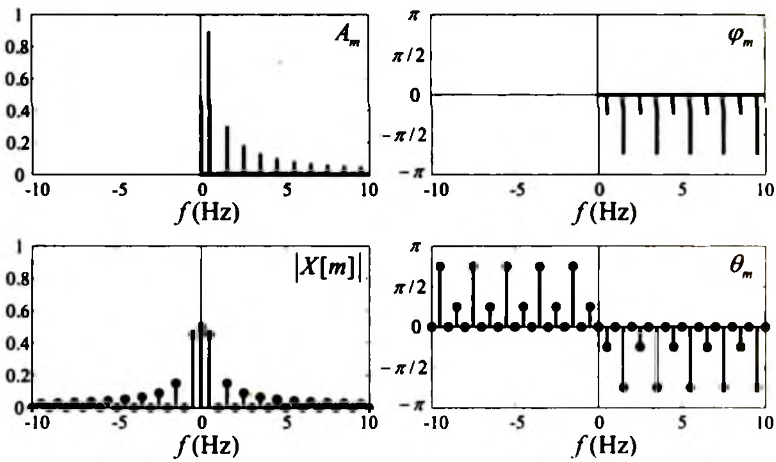
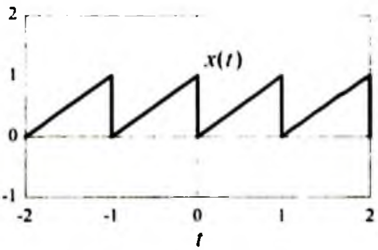


Figure 7.5: Amplitude and phase spectra of the model in Figure 7.1. The fundamental period T_0 of the model is 2, and thus the frequency interval Δf is 0.5.

7.4.2 Even and Odd Functions

Example 7.9 Determine the Fourier series coefficients $X[m]$ of the model shown below.



Hint: Refer to expression B.44.

Solution

With $T_0 = 1$, $\Omega = 2\pi$, we take the integration range between 0 and T_0.

$$X[m] = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jm\Omega t}\, dt = \int_0^1 te^{-j2m\pi t}\, dt.$$

$$= \left[\frac{te^{-j2m\pi t}}{(-j2m\pi)} - \frac{e^{-j2m\pi t}}{(-j2m\pi)^2}\right]_0^1 = \left[\frac{jte^{-j2m\pi t}}{2m\pi} + \frac{e^{-j2m\pi t}}{(2m\pi)^2}\right]_0^1$$

$$= \frac{je^{-j2m\pi}}{2m\pi} + \frac{e^{-j2m\pi} - 1}{(2m\pi)^2} = \frac{j}{2m\pi}$$

The above expression is, however, not suitable for $m = 0$. We thus separately evaluate $X[0]$ as follows:

$$X[0] = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt = \int_0^1 t\, dt = \frac{1}{2}.$$

Figure 7.6 shows the amplitude and phase spectra of the periodic time function in Example 7.9. It is worthwhile to note that except at zero fre-

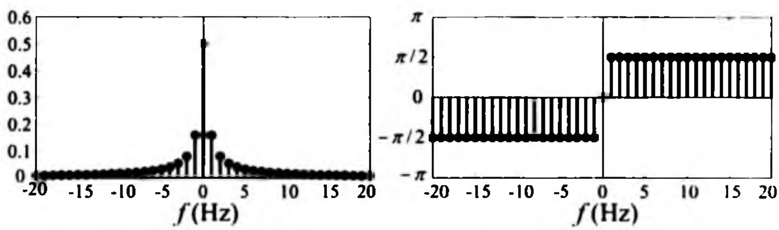


Figure 7.6: Amplitude and phase spectra of the model in Example 7.9. The fundamental period T_0 of the model is 1, and thus the frequency interval Δf is 1.

quency, phase values are either $\pi/2$ or $-\pi/2$. The reason is mathematically explained by analyzing $X[m]$ as

$$X[m] = \frac{j}{2m\pi} = \frac{1}{2m\pi}\, e^{j\pi/2}, \quad |X[m]| = \frac{1}{2|m|\pi},$$

and

$$\theta_m = \begin{cases} \pi/2 & (m > 0), \\ -\pi/2 & (m < 0). \end{cases}$$

We should not, however, satisfy without having geometrical meaning of the above mathematical explanation. Recall that

$$\cos(m\Omega t \pm \pi/2) = \mp \sin(m\Omega t).$$

In other words, exhibiting $\pm\pi/2$ in the phase spectrum means that the Fourier series only involves sine functions. Ignoring the zero frequency component, the original periodic function $x(t)$ is thus an odd function. It is evident in Example 7.9 that amplitude shifting the model $(x(t) - 1/2)$ yields an odd function.

Having the expression of $X[m]$, one can reconstruct the original periodic function with expression 7.20. It is, however, inevitable to limit the range of summation as follows:

$$x(t) = \sum_{m=-k}^{k} X[m]\, e^{jm\Omega t}. \tag{7.23}$$

Expression 7.23 is called the *truncated Fourier series*. Note that the range of summation should be symmetric around $m = 0$, because losing the

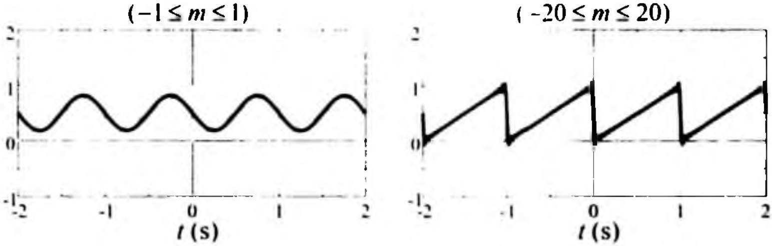
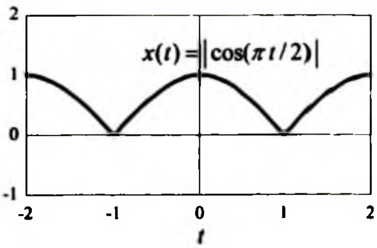


Figure 7.7: Truncated Fourier series of the model in Example 7.9

symmetry yields complex-valued $x(t)$, which is, of course, unrealistic. Figure 7.7 shows reconstructed models. It is clear that the truncated Fourier series well reconstructs the model in Example 7.9.

Example 7.10 Determine the Fourier series coefficients $X[m]$ of the model shown below.



Solution

With $T_0 = 2$, $\Omega = \pi$, we take the integration range between $-T_0/2$ and $T_0/2$.

$$X[m] = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jm\Omega t}\, dt = \frac{1}{2}\int_{-1}^{1} \cos(\pi t/2)\, e^{-jm\pi t}\, dt$$

$$= \frac{1}{4}\int_{-1}^{1} [e^{j\pi t/2} + e^{-j\pi t/2}]\, e^{-jm\pi t}\, dt$$

$$= \frac{1}{4}\int_{-1}^{1} e^{j(\pi/2-m\pi)t}\, dt + \frac{1}{4}\int_{-1}^{1} e^{-j(\pi/2+m\pi)t}\, dt$$

$$= \left[\frac{e^{j(\pi/2-m\pi)t}}{j4(\pi/2-m\pi)} \right]_{-1}^{1} - \left[\frac{e^{-j(\pi/2+m\pi)t}}{j4(\pi/2+m\pi)} \right]_{-1}^{1}$$

$$= \frac{e^{j(\pi/2-m\pi)} - e^{-j(\pi/2-m\pi)}}{j4(\pi/2-m\pi)} - \frac{e^{-j(\pi/2+m\pi)} - e^{j(\pi/2+m\pi)}}{j4(\pi/2+m\pi)}$$

$$= \frac{e^{j\pi/2}\,e^{-jm\pi} - e^{-j\pi/2}\,e^{jm\pi}}{j4(\pi/2-m\pi)} - \frac{e^{-j\pi/2}\,e^{-jm\pi} - e^{j\pi/2}\,e^{jm\pi}}{j4(\pi/2+m\pi)}$$

$$= \frac{e^{-jm\pi} + e^{jm\pi}}{4(\pi/2-m\pi)} + \frac{e^{-jm\pi} + e^{jm\pi}}{4(\pi/2+m\pi)} = \frac{\cos(m\pi)}{\pi(1-2m)} + \frac{\cos(m\pi)}{\pi(1+2m)}.$$

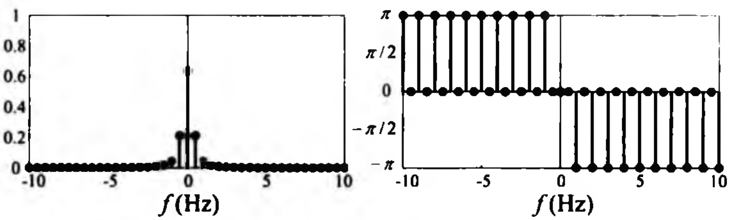


Figure 7.8: Amplitude and phase spectra of the model in Example 7.10. The fundamental period T_0 of the model is 2, and thus the frequency interval Δf is 0.5.

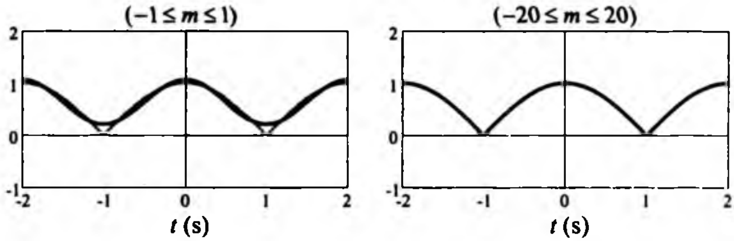


Figure 7.9: Truncated Fourier series of the model in Example 7.10

The periodic time function in Example 7.10 has an even symmetry. And the even symmetry enforces the Fourier series coefficient $X[m]$ to be real, regardless of the value of m. The resultant amplitude and phase spectra are shown in Figure 7.8. Note that the phase values of an even function must be either 0 or $\pm\pi$.

PROBLEMS

Problem 7.1 Determine the fundamental frequency and Fourier series coefficients a_m and b_m of the following periodic function:

$$x(t) = 2 + \cos(2\pi t/3) + 4\sin(5\pi t/3).$$

Problem 7.2 Determine the fundamental frequency and Fourier series coefficients a_m and b_m of the following Fourier series expression:

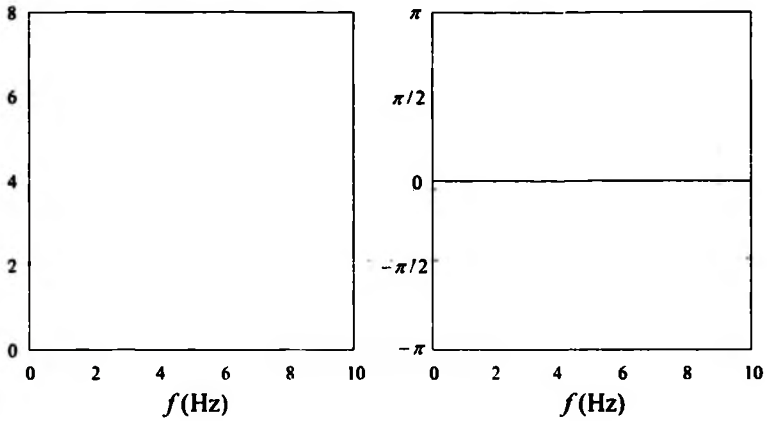
$$x(t) = \sum_{m=1}^{\infty} \frac{2}{m} \sin(4m\pi t)\cos^2(m\pi).$$

Problem 7.3 Use the result of Example 7.4 and make a MATLAB script that synthesizes the periodic function of Example 7.4. You may modify the script of Example 7.1.

Problem 7.4 Use the result of Example 7.5 and make a MATLAB script that synthesizes the periodic function of Example 7.5. You may modify the script of Example 7.1.

Problem 7.5 Sketch the amplitude and phase spectra of the following periodic function.

$$x(t) = 6\sin(2\pi t - \pi/6) - 4\cos(4\pi t) + 3\cos(6\pi t + 2\pi/3).$$

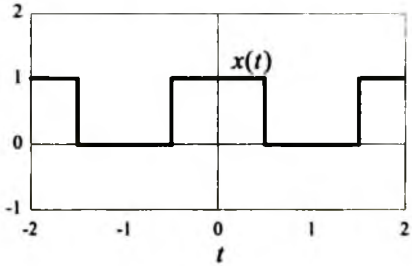


Problem 7.6 Obtain the exponential Fourier series coefficients for the following periodic function:

$$x(t) = 1 + \cos^2(\pi t) - \sin^3(\pi t).$$

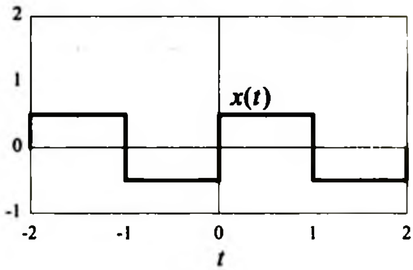
Problem 7.7 Show that the exponential Fourier series coefficient $X[m]$ of the periodic function shown below is

$$X[m] = \frac{j[(-j)^m - (j)^m]}{2m\pi}.$$

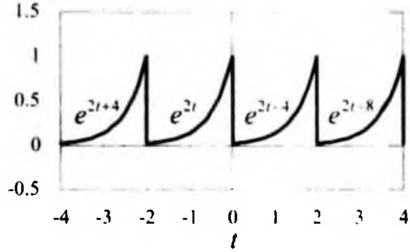


Problem 7.8 Show that the exponential Fourier series coefficient $X[m]$ of the periodic function shown below is

$$X[m] = \frac{j[-1 + (-1)^m]}{2m\pi}.$$



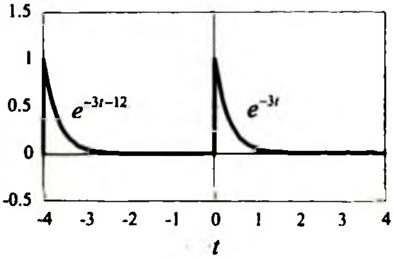
Problem 7.9 Which of the following is the exponential Fourier series coefficient $X[m]$ of the periodic function shown below? Choose one.



a. $\dfrac{1-e^{-4}}{4+jm\pi}$ b. $\dfrac{1-e^{-4}}{4-jm\pi}$ c. $\dfrac{1-e^{-4}}{4+j2m\pi}$ d. $\dfrac{1-e^{-4}}{4-j2m\pi}$

e. $\dfrac{1-e^{-4}}{4+j3m\pi}$ f. $\dfrac{1-e^{-4}}{4-j3m\pi}$ g. $\dfrac{1-e^{-4}}{4+j4m\pi}$ h. $\dfrac{1-e^{-4}}{4-j4m\pi}$

Problem 7.10 Which of the following is the exponential Fourier series coefficient $X[m]$ of the periodic function shown below? Choose one.



a. $\dfrac{1-e^{-12}}{12+jm\pi}$ b. $\dfrac{1-e^{-12}}{12-jm\pi}$ c. $\dfrac{1-e^{-12}}{12+j2m\pi}$ d. $\dfrac{1-e^{-12}}{12-j2m\pi}$

e. $\dfrac{1-e^{-12}}{12+j3m\pi}$ f. $\dfrac{1-e^{-12}}{12-j3m\pi}$ g. $\dfrac{1-e^{-12}}{12+j4m\pi}$ h. $\dfrac{1-e^{-12}}{12-j4m\pi}$

Chapter 8

PROPERTIES OF FOURIER SERIES

8.1 LINEARITY OF FOURIER SERIES

Consider Fourier series expansion of two periodic time functions $x_1(t)$ and $x_2(t)$. We assume the two functions have an identical fundamental frequency Ω such that

$$x_1(t) = \sum_{m=-\infty}^{\infty} X_1[m]\, e^{jm\Omega t} \quad \text{and} \quad x_2(t) = \sum_{m=-\infty}^{\infty} X_2[m]\, e^{jm\Omega t}. \tag{8.1}$$

It is obvious that a linear combination of the two functions:

$$x_3(t) = ax_1(t) + bx_2(t) \tag{8.2}$$

also yields a periodic function, and the fundamental frequency of $x_3(t)$ must be Ω. We thus express $x_3(t)$ as

$$x_3(t) = \sum_{m=-\infty}^{\infty} X_3[m]\, e^{jm\Omega t} \tag{8.3}$$

and the Fourier series coefficient as

$$X_3[m] = aX_1[m] + bX_2[m]. \tag{8.4}$$

The above expression represents the linearity of the Fourier series. Note that the linearity is easily extended to an arbitrary number of functions as far as all of those functions have an identical fundamental frequency.

Three periodic functions shown in Figure 8.1 have the identical fundamental frequency ($\Omega = 2\pi/T_0 = \pi$) and satisfy the following linear relation:

$$x_3(t) = x_1(t) + x_2(t).$$

The Fourier series coefficients $X_1[m]$ and $X_2[m]$ have been derived in Problems 7.7 and 7.8 as

$$X_1[m] = \frac{j[(-j)^m - (j)^m]}{2m\pi}, \tag{8.5}$$

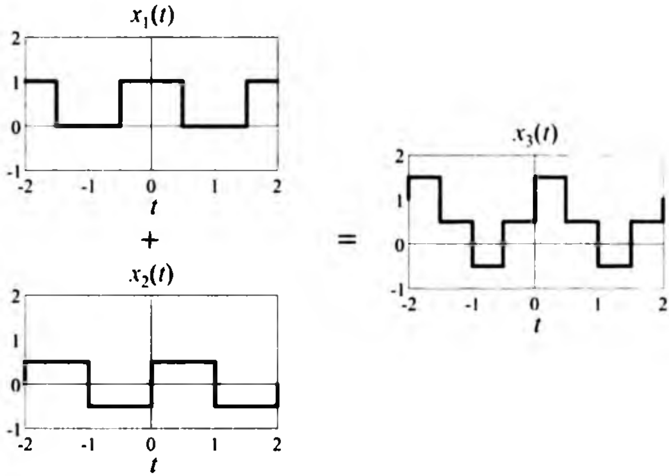


Figure 8.1: Examples of periodic functions that have an identical fundamental frequency

and

$$X_2[m] = \frac{j[-1 + (-1)^m]}{2m\pi}. \tag{8.6}$$

The Fourier series coefficient $X_3[m]$ has been derived in Example 7.8 as

$$X_3[m] = \frac{j[-1 + (-j)^m + (-1)^m - (j)^m]}{2m\pi}.$$

It is obvious that the above Fourier series coefficients satisfy the following relation:

$$X_3[m] = X_1[m] + X_2[m].$$

In other words, Fourier series coefficients satisfy the same linear relation of the periodic functions the Fourier series coefficients are representing.

8.2 FOURIER SERIES AND EVEN / ODD SYMMETRIES

We have discussed in Chapter 7 that the Fourier series expansion of an even periodic function only requires cosine functions and, as a result of that, the phase values must be either 0 or $\pm\pi$. While handling an odd periodic function, on the other hand, sine functions are enough for the

Fourier series expansion. and the phase spectrum should only exhibit 0 or $\pm\pi/2$ values.

One can associate the Fourier series coefficient $X[m]$ with the even/odd symmetry of periodic functions. Consider the following Fourier series expansion:

$$\begin{aligned} x(t) &= \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t} \\ &= \cdots + X[-2]\, e^{-j2\Omega t} + X[-1]\, e^{-j\Omega t} + X[0] \\ &\quad + X[1]\, e^{j\Omega t} + X[2]\, e^{j2\Omega t} + \cdots . \end{aligned}$$

The Fourier series expansion of $x(-t)$ is then expressed as

$$\begin{aligned} x(-t) &= \sum_{m=-\infty}^{\infty} X[m]\, e^{-jm\Omega t} \\ &= \cdots + X[-2]\, e^{j2\Omega t} + X[-1]\, e^{j\Omega t} + X[0] \\ &\quad + X[1]\, e^{-j\Omega t} + X[2]\, e^{-j2\Omega t} + \cdots . \end{aligned}$$

Comparing the two expressions, we find that the even symmetry of a periodic time function (i.e., $x_e(t) = x_e(-t)$) demands

$$X_e[m] = X_e[-m].$$

We now consider expression 7.19 and rewrite the above expression as

$$X_e[m] = X_e^*[m], \tag{8.7}$$

where the $*$ symbol denotes the *complex conjugate*. Expression 8.7 thus means that the Fourier series coefficients of an even function must be real. In the same token, the Fourier series coefficients of an odd function must be zero or imaginary such that

$$X_o[m] = -X_o^*[m]. \tag{8.8}$$

Figures 8.2 and 8.3 show the amplitude and phase spectra of $x_1(t)$ and $x_2(t)$ in Figure 8.1, respectively. It is clear that the even symmetry of $x_1(t)$ yields real-valued $X_1[m]$ (expression 8.5), and the real-valued $X_1[m]$, in turn, accompanies the phase spectrum that only exhibits 0 or $\pm\pi$. Similarly, the odd symmetry of $x_2(t)$ yields imaginary-valued $X_2[m]$ (expression 8.6), and its corresponding phase spectrum only exhibits 0 or $\pm\pi/2$. Incidentally, it is noteworthy that amplitude spectra always have even symmetry, but phase spectra always demonstrate odd symmetry.

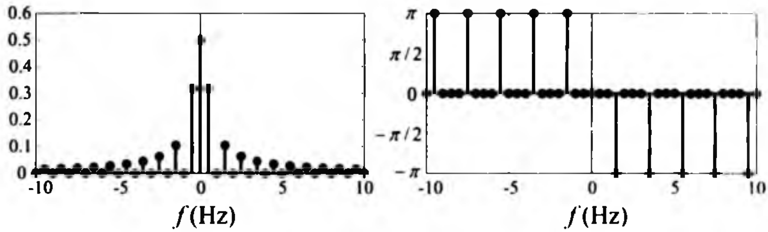


Figure 8.2: Amplitude and phase spectra of the even function $x_1(t)$ in Figure 8.1. The fundamental period T_0 of the function is 2, and thus the frequency interval Δf is 0.5.

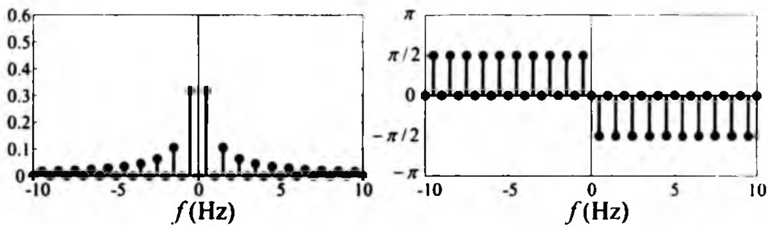


Figure 8.3: Amplitude and phase spectra of the odd function $x_2(t)$ in Figure 8.1. The fundamental period T_0 of the function is 2, and thus the frequency interval Δf is 0.5.

8.3 FOURIER SERIES AND TIME OPERATIONS

Several time operations have been introduced in Chapter 3. We consider those time operations again and discuss how each different time operations of a system alter frequency characteristics of periodic time functions.

8.3.1 Time Shifting

Time shifting is described by the following input-output relation:

$$y(t) = x(t - t_0),$$

and the fundamental frequency of the input function is preserved. We thus write the following expressions:

$$x(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t} \quad \text{and} \quad y(t) = \sum_{m=-\infty}^{\infty} Y[m]\, e^{jm\Omega t},$$

and associate the two as

$$y(t) = \sum_{m=-\infty}^{\infty} Y[m]\, e^{jm\Omega t} = x(t - t_0) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega(t-t_0)}$$
$$= \sum_{m=-\infty}^{\infty} X[m]\, e^{-jm\Omega t_0}\, e^{jm\Omega t}.$$

It is evident from the above expression that

$$Y[m] = X[m]\, e^{-jm\Omega t_0}. \tag{8.9}$$

Denoting phase spectra of $x(t)$ and $y(t)$ as θ_m and ψ_m such that

$$X[m] = |X[m]|\, e^{j\theta_m} \quad \text{and} \quad Y[m] = |Y[m]|\, e^{j\psi_m},$$

we write

$$|Y[m]| = |X[m]|,$$

and

$$\psi_m = \theta_m - m\Omega t_0. \tag{8.10}$$

In other words, time shifting operation does not alter the amplitude spectrum of the input function but tremendously changes the phase spectrum.

Figure 8.4 shows an example of time shifting operation with $t_0 = 1/12$. The Fourier series coefficient $X[m]$ is given in Example 7.9 as

$$X[m] = \begin{cases} 1/2 & (m = 0), \\ e^{j\pi/2}/(2m\pi) & (m \neq 0), \end{cases}$$

and the amplitude spectrum $|X[m]|$ and phase spectrum θ_m are expressed as follows:

$$|X[m]| = \begin{cases} 1/(2m\pi) & (m > 0) \\ 1/2 & (m = 0) \\ 1/(2|m|\pi) & (m < 0) \end{cases} \quad \text{and} \quad \theta_m = \begin{cases} \pi/2 & (m > 0) \\ 0 & (m = 0) \\ -\pi/2 & (m < 0) \end{cases}$$

Applying $T_0 = 1$, $\Omega = 2\pi$ and $t_0 = 1/12$ to expressions 8.9 and 8.10, we derive the Fourier series coefficient $Y[m]$ that correspond to the periodic time function $y(t)$ in Figure 8.4 as

$$Y[m] = \begin{cases} 1/2 & (m = 0), \\ e^{-j(m\pi/6-\pi/2)}/(2m\pi) & (m \neq 0). \end{cases}$$

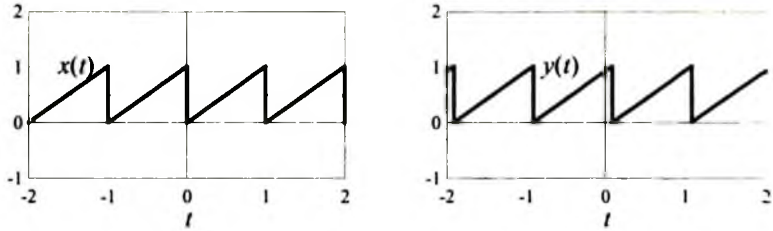


Figure 8.4: Periodic time functions before and after the time shifting: $y(t) = x(t - 1/12)$.

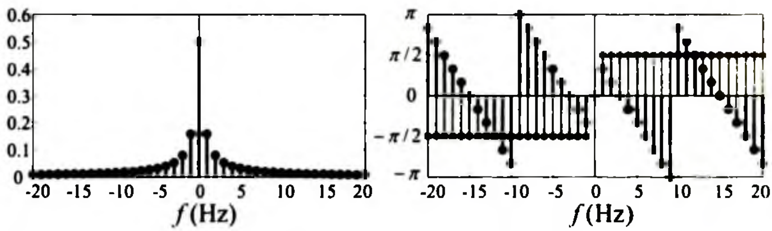


Figure 8.5: Time shifting and its influence on the amplitude and phase spectra. Gray and black plots are related to $x(t)$ and $y(t)$ in Figure 8.4, respectively. The fundamental period T_0 of the two functions is 1, and thus the frequency interval Δf is 1.

The amplitude spectrum $|Y[m]|$ and phase spectrum ψ_m are thus expressed as follows:

$$|Y[m]| = \begin{cases} 1/(2m\pi) & (m > 0) \\ 1/2 & (m = 0) \\ 1/(2|m|\pi) & (m < 0) \end{cases} \quad \text{and} \quad \psi_m = \begin{cases} \pi/2 - m\pi/6 & (m > 0) \\ 0 & (m = 0) \\ -\pi/2 - m\pi/6 & (m < 0) \end{cases}$$

Note that the above approach of deriving the Fourier series coefficient $Y[m]$ is much simpler than the direct integration of $y(t)$ via expression 7.21. The amplitude and phase spectra of $x(t)$ and $y(t)$ are shown in Figure 8.5.

8.3.2 Time Reversal

Time reversal is described by the following input-output relation:

$$y(t) = x(-t),$$

and the fundamental frequency of the input function is preserved. We thus write the following expressions:

$$x(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t} \quad \text{and} \quad y(t) = \sum_{m=-\infty}^{\infty} Y[m]\, e^{jm\Omega t},$$

and associate the two as

$$y(t) = \sum_{m=-\infty}^{\infty} Y[m]\, e^{jm\Omega t} = x(-t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{-jm\Omega t}$$

$$= \sum_{m=-\infty}^{\infty} X[-m]\, e^{jm\Omega t}.$$

It is evident from the above expression that

$$Y[m] = X[-m] = X^*[m]. \tag{8.11}$$

Denoting phase spectra of $x(t)$ and $y(t)$ as θ_m and ψ_m such that

$$X[m] = |X[m]|\, e^{j\theta_m} \quad \text{and} \quad Y[m] = |Y[m]|\, e^{j\psi_m},$$

we write

$$|Y[m]| = |X[m]|,$$

and

$$\psi_m = -\theta_m. \tag{8.12}$$

In other words, time reversal operation does not alter the amplitude spectrum of the input function but changes the sign of the phase spectrum.

Figure 8.6 shows an example of time reversal operation. The Fourier series coefficient $Y[m]$ is derived as

$$Y[m] = \begin{cases} 1/2 & (m = 0), \\ e^{-j\pi/2}/(2m\pi) & (m \neq 0), \end{cases}$$

and the amplitude spectrum $|Y[m]|$ and phase spectrum ψ_m are expressed as follows:

$$|Y[m]| = \begin{cases} 1/(2m\pi) & (m > 0) \\ 1/2 & (m = 0) \\ 1/(2|m|\pi) & (m < 0) \end{cases} \quad \text{and} \quad \psi_m = \begin{cases} -\pi/2 & (m > 0) \\ 0 & (m = 0) \\ \pi/2 & (m < 0) \end{cases}$$

The amplitude and phase spectra of $x(t)$ and $y(t)$ are shown in Figure 8.7.

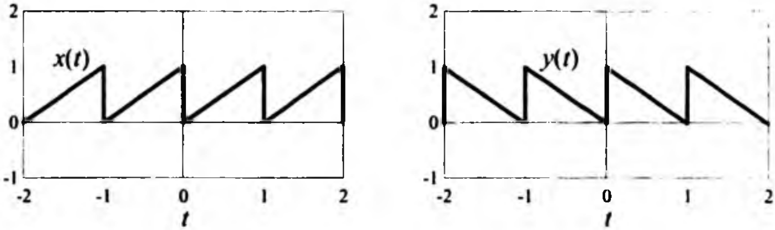


Figure 8.6: Periodic time functions before and after the time reversal: $y(t) = x(-t)$.

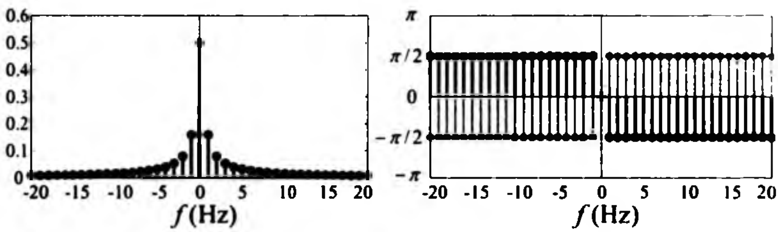
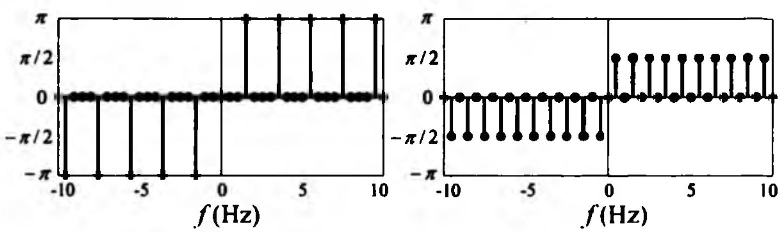


Figure 8.7: Time reversal and its influence on the amplitude and phase spectra. Gray and black plots are related to $x(t)$ and $y(t)$ in Figure 8.6, respectively. The fundamental period T_0 of the two functions is 1, and thus the frequency interval Δf is 1.

Example 8.1 The phase spectrua of $x_1(t)$ and $x_2(t)$ in Figure 8.1 are shown in Figures 8.2 and 8.3. Sketch the phase spectra of $y_1(t) = x_1(-t)$ and $y_2(t) = x_2(-t)$.

Solution



Note that $x_1(t)$ in Figure 8.1 is an even function and time reversal should not alter the function such that $y_1(t) = x_1(-t) = x_1(t)$. However, phase spectra of $x_1(t)$ and $y_1(t)$ look different (compare Example 8.1 and Figure 8.2). Do not be puzzled by the apparent difference. The two phase spectra may look different, but they are, in fact, identical to each other because phase value π is identical to $-\pi$ within a phase spectrum.

8.3.3 Time Scaling

Time scaling is described by the following input-output relation:

$$y(t) = x(at) \quad (a > 0),$$

and unlike time shifting or reversal, the fundamental frequency of the input function is NOT preserved. Assuming the following input function:

$$x(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t},$$

we can express the output function as

$$y(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega a t}.$$

It is evident from the above expression that the fundamental frequency of $y(t)$ is different from that of $x(t)$. Denoting the fundamental frequency of $y(t)$ as $\tilde{\Omega}$ and the fundamental period as $\tilde{T}_0$, we may establish the following relations:

$$\tilde{\Omega} = \Omega a, \quad (8.13)$$

$$\tilde{T}_0 = T_0/a, \quad (8.14)$$

and express the Fourier series expansion of $y(t)$ as

$$y(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\tilde{\Omega} t}.$$

The above expression demonstrates that time scaling operation does not alter the formal representation of the Fourier series expansion, and the

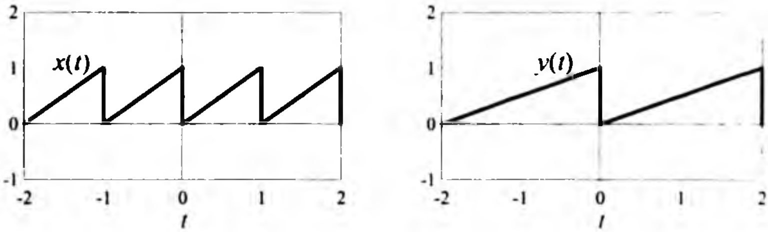


Figure 8.8: Periodic time functions before and after the time scaling: $y(t) = x(t/2)$.

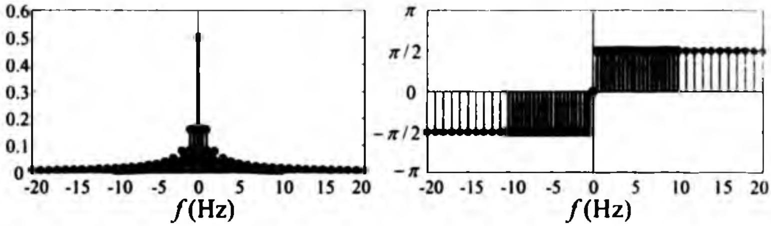


Figure 8.9: Time scaling and its influence on the amplitude and phase spectra. Gray and black plots are related to $x(t)$ and $y(t)$ in Figure 8.8, respectively. The fundamental periods of $x(t)$ and $y(t)$ are 1 and 2, respectively, and thus the frequency intervals Δf are 1 and 0.5, respectively.

only difference the time scaling operation makes is the change of the fundamental frequency.

It is noteworthy that time scaling is related to *frequency scaling* in a systematic way. Expression 8.14 indicates that time compression ($a > 1$) reduces the fundamental period $\tilde{T}_0$ and thus increases Δf in the frequency domain. On the other hand, time expansion ($a < 1$) enlarges the fundamental period of $y(t)$ and, as a result of that, decreases Δf in the frequency domain. In other words, time compression invokes frequency expansion, whereas time expansion accompanies frequency compression. Figure 8.8 shows an example of time expansion operation, and the amplitude and phase spectra of $x(t)$ and $y(t)$ are shown in Figure 8.9.

8.3.4 Time Differentiation

Time differentiation is described by the following input-output relation:

$$y(t) = \frac{dx(t)}{dt},$$

and the fundamental frequency of the input function is preserved. We thus write the following expressions:

$$x(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t} \quad \text{and} \quad y(t) = \sum_{m=-\infty}^{\infty} Y[m]\, e^{jm\Omega t},$$

and associate the two as

$$y(t) = \sum_{m=-\infty}^{\infty} Y[m]\, e^{jm\Omega t} = \frac{dx(t)}{dt} = \sum_{m=-\infty}^{\infty} jm\Omega X[m]\, e^{jm\Omega t}.$$

It is evident from the above expression that

$$Y[m] = jm\Omega X[m] = m\Omega X[m]\, e^{j\pi/2}. \tag{8.15}$$

Denoting phase spectra of $x(t)$ and $y(t)$ as θ_m and ψ_m such that

$$X[m] = |X[m]|\, e^{j\theta_m} \quad \text{and} \quad Y[m] = |Y[m]|\, e^{j\psi_m},$$

we write

$$|Y[m]| = |m\Omega X[m]|,$$

and

$$\psi_m = \begin{cases} \theta_m + \pi/2 & (m > 0)\,, \\ 0 & (m = 0)\,, \\ \theta_m - \pi/2 & (m < 0)\,. \end{cases} \tag{8.16}$$

In other words, time differentiation operation alters both the amplitude and phase spectra and, more interestingly, involves ±90° phase shifts in the phase spectrum.

Consider, for example, the time differentiation operation depicted in Figure 8.10. The Fourier series coefficient $X[m]$ is given in Example 7.10 as

$$X[m] = \frac{\cos(m\pi)}{\pi(1 - 2m)} + \frac{\cos(m\pi)}{\pi(1 + 2m)}.$$

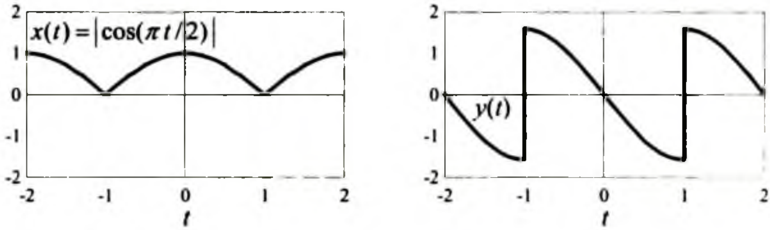


Figure 8.10: Periodic time functions before and after the time differentiation: $y(t) = dx(t)/dt$.

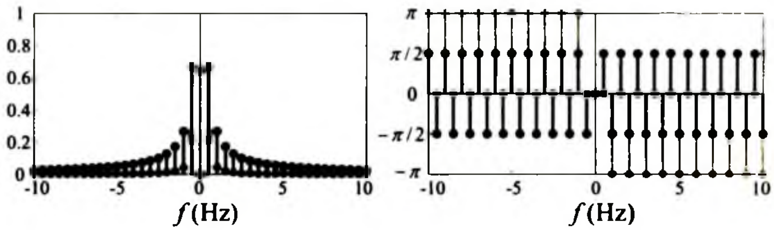


Figure 8.11: Time differentiation and its influence on the amplitude and phase spectra. Gray and black plots are related to $x(t)$ and $y(t)$ in Figure 8.10, respectively. The fundamental period T_0 of the two functions is 2, and thus the frequency interval Δf is 0.5.

It is obvious that regardless of the value of m, $X[m]$ is always real and changes sign in an alternating fashion as m varies. Applying $T_0 = 2$ and $\Omega = \pi$ to expression 8.15, we derive the Fourier series coefficient $Y[m]$ that correspond to the periodic time function $y(t)$ in Figure 8.10 as

$$Y[m] = m\pi \left[\frac{\cos(m\pi)}{\pi(1-2m)} + \frac{\cos(m\pi)}{\pi(1+2m)} \right] e^{j\pi/2}.$$

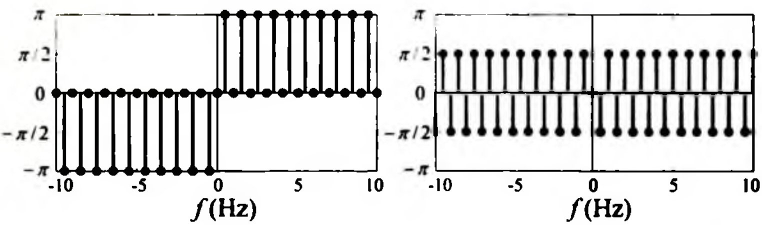
Note that $y(t)$ has an odd symmetry whereas $x(t)$ has an even symmetry, and, as a result of that, $Y[m]$ is always imaginary (for $m \neq 0$) and phase values of the coefficients are $\pm\pi/2$. The amplitude and phase spectra of $x(t)$ and $y(t)$ are shown in Figure 8.11. One should observe 90° phase shift for $f > 0$ and −90° phase shift for $f < 0$.

Example 8.2 Consider $x(t)$ shown in Figure 8.10 and the following functions:

$$y_1(t) = \frac{d^2x(t)}{dt^2}, \quad \text{and} \quad y_2(t) = \frac{d^3x(t)}{dt^3}.$$

Sketch the phase spectrum of $y_1(t)$ and $y_2(t)$.

Solution



8.4 PARSEVAL'S THEOREM

We have introduced in Chapter 2 the concept of average power P of a continuous-time signal and described the average power as expression 2.22. *Parseval's theorem* states that if $x(t)$ is a periodic function with fundamental period T_0, then the *average power* P of the signal is defined by

$$P = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt. \tag{8.17}$$

Applying the Fourier series expansion to the periodic function as

$$x(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t} \quad \text{with} \quad X[m] = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jm\Omega t}\, dt,$$

one can write the following relation:

$$|x(t)|^2 = x(t)\,x(t) = \left[\sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t}\right] x(t) = \sum_{m=-\infty}^{\infty} X[m]\, x(t)\, e^{jm\Omega t}.$$

The average power P is thus expressed as

$$P = \frac{1}{T_0} \int_0^{T_0} \left[\sum_{m=-\infty}^{\infty} X[m]\, x(t)\, e^{jm\Omega t} \right] dt$$
$$= \sum_{m=-\infty}^{\infty} X[m] \left[\frac{1}{T_0} \int_0^{T_0} x(t)\, e^{jm\Omega t}\, dt \right].$$

Note that the integral at the end of above expression can be substantially simplified as

$$\frac{1}{T_0} \int_0^{T_0} x(t)\, e^{jm\Omega t}\, dt = X^*[m].$$

In other words, we can express the average power of a periodic function in terms of the Fourier series coefficients as follows:

$$P = \sum_{m=-\infty}^{\infty} X[m]\, X^*[m] = \sum_{m=-\infty}^{\infty} |X[m]|^2. \tag{8.18}$$

Example 8.3 Consider $x(t)$ shown in Figure 8.4 and derive the average power P of the periodic function via expression 8.17.

Solution

$$P = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2\, dt = \int_0^1 x^2\, dt = \frac{1}{3}.$$

Example 8.4 Consider $x(t)$ shown in Figure 8.4 and derive the average power P of the periodic function via expression 8.18.
Hint: Refer to expression B.50.

Solution

$$X[m] = \begin{cases} 1/2 & (m = 0)\,, \\ e^{j\pi/2}/(2m\pi) & (m \neq 0)\,, \end{cases}$$

$$P = \sum_{m=-\infty}^{\infty} |X[m]|^2$$

$$= |X[0]|^2 + \sum_{m=1}^{\infty} |X[m]|^2 + \sum_{m=-1}^{-\infty} |X[m]|^2$$

$$= \frac{1}{4} + \sum_{m=1}^{\infty} \frac{1}{(2m\pi)^2} + \sum_{m=-1}^{-\infty} \frac{1}{(2m\pi)^2}$$

$$= \frac{1}{4} + 2\sum_{m=1}^{\infty} \frac{1}{(2m\pi)^2} = \frac{1}{4} + \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{1}{3}.$$

8.5 FOURIER SERIES AND LTI SYSTEMS

In Chapter 5, we have discussed the concept of *linear time-invariant* (LTI) systems (Figure 5.5) and described the input-output relation of an LTI system as

$$y(t) = x(t) * h(t),$$

where $h(t)$ denotes the impulse response of the LTI system. Assuming a complex exponential input function as follows:

$$x(t) = e^{j\omega t},$$

we express the output from the LTI system as

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}\, h(\tau)\, d\tau$$

$$= \left[\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau\right] e^{j\omega t}.$$

The integral in the last term of the above expression is called the *frequency response* of the LTI system. In other words, we define the frequency response $H(\omega)$ of an LTI system as

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt, \qquad (8.19)$$

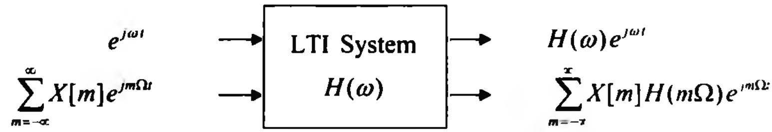


Figure 8.12: Concept of frequency response $H(\omega)$

and establish the following input-output relation:

$$x(t) = e^{j\omega t} \quad \longrightarrow \quad y(t) = H(\omega)\, e^{j\omega t}. \tag{8.20}$$

Combined with the concept of frequency response, Fourier series expansion enables one to conveniently analyze an LTI system. Given the frequency response $H(\omega)$ of the system, we write the following input-output relation:

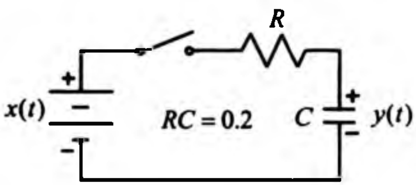
$$x(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t} \longrightarrow y(t) = \sum_{m=-\infty}^{\infty} X[m]\, H(m\Omega)\, e^{jm\Omega t}, \tag{8.21}$$

and associate the Fourier series coefficients as

$$X[m] \quad \longrightarrow \quad Y[m] = X[m]\, H(m\Omega). \tag{8.22}$$

The concept of frequency response of an LTI system and its relation with the Fourier series is summarized in Figure 8.12.

Example 8.5 Consider an *RC circuit* shown below.



The *RC time constant* of the circuit is 0.2. Regard the battery and capacitor voltages as the input $x(t)$ and output $y(t)$ of the circuit. What is the frequency response of the system?

Solution

Referring expression 5.12, we write the impulse response of the LTI system as

$$h(t) = \frac{1}{RC} e^{-t/RC}\, u(t) = 5e^{-5t}\, u(t),$$

and derive the frequency response of the system as follows:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt = 5 \int_{-\infty}^{\infty} e^{-5t}\, u(t)\, e^{-j\omega t}\, dt$$

$$= 5 \int_{0}^{\infty} e^{-(5+j\omega)t}\, dt = \frac{5}{5+j\omega}.$$

The concept of frequency response is useful for understanding an RC circuit. Consider, for example, the circuit in Example 8.5. We have studied in Chapter 5 that the RC circuit satisfies the input-output relation shown in Figure 8.13. Considering that $x(t)$ in Figure 8.13 is an amplitude shifted function of $x_2(t)$ in Figure 8.1, we express the Fourier series coefficient of $x(t)$ as

$$X[m] = \frac{j[-1+(-1)^m]}{2m\pi} \quad (m \neq 0),$$

and $X[0] = 1/2$. Expression 8.22 and Example 8.5 enable one to describe the Fourier series coefficient of $y(t)$ as

$$Y[m] = X[m]\, H(m\Omega) = \frac{j[-1+(-1)^m]}{2m\pi} \frac{5}{5+jm\pi} \quad (m \neq 0),$$

and $Y[0] = 1/2$. Note that the concept of the frequency response greatly simplifies the derivation of the Fourier series coefficients $Y[m]$. The amplitude and phase spectra of the input and output functions in Figure 8.13 are presented in Figure 8.14. Truncated Fourier series are also illustrated in Figure 8.15. It is evident that the Fourier series well reconstructs the input and output functions of the RC circuit.

It has been shown that we can understand the frequency characteristics of an LTI system by analyzing the amplitude and phase spectra of the

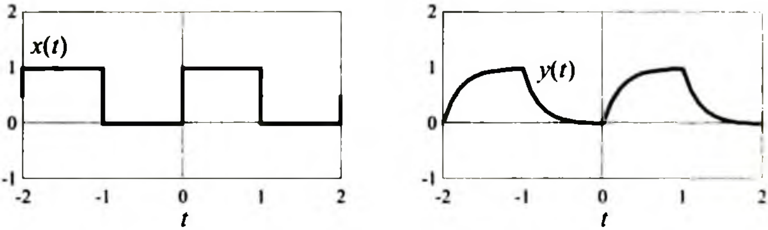


Figure 8.13: Input and output functions of the RC circuit in Example 8.5. The fundamental period T_0 of the two functions are 2, and the fundamental frequency Ω is thus π.

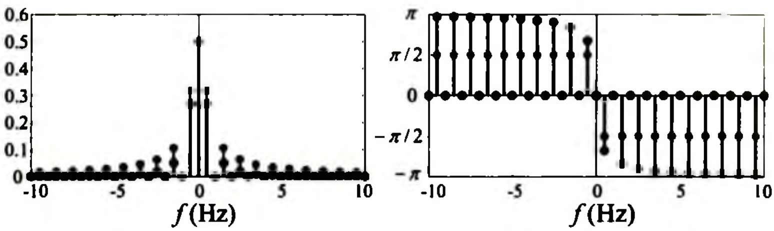


Figure 8.14: RC circuit and its influence on the amplitude and phase spectra. Gray and black plots are related to $x(t)$ and $y(t)$ in Figure 8.13, respectively. The fundamental period T_0 of the two functions is 2, and thus the frequency interval Δf is 0.5.

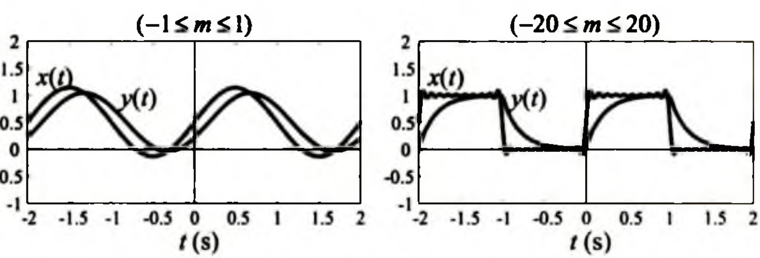


Figure 8.15: Truncated Fourier series that assess input and output functions shown in Figure 8.13. Gray and black plots are related to $x(t)$ and $y(t)$, respectively.

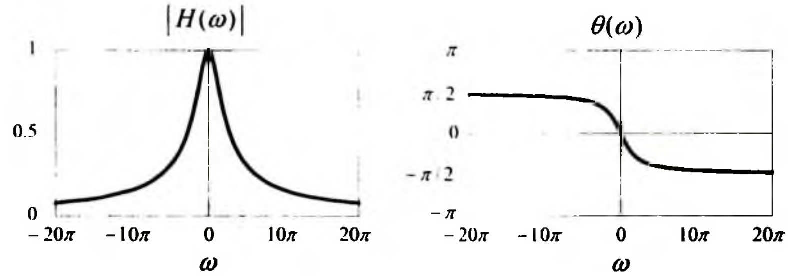


Figure 8.16: The amplitude and phase spectrum of the frequency response derived in Example 8.5

input and output functions. There is, in fact, a more convenient approach of understanding the frequency characteristics of the system. That is to analyze the amplitude and phase spectra of the frequency response itself, instead of the input and output functions. In Example 8.5, we have seen that the frequency response of the RC circuit is

$$H(\omega) = \frac{5}{5 + j\omega}.$$

Denoting the amplitude and phase of the frequency response as $|H(\omega)|$ and $\theta(\omega)$, respectively, we can show that

$$|H(\omega)| = \frac{5}{\sqrt{25 + \omega^2}},$$

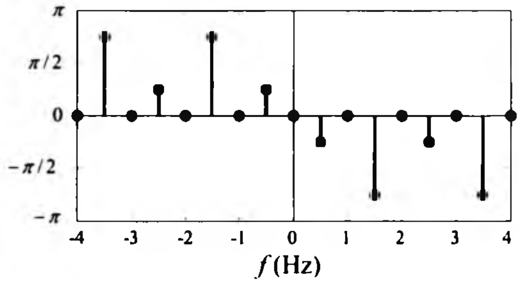
and

$$\theta(\omega) = -\tan^{-1}\left(\frac{\omega}{5}\right).$$

The amplitude and phase spectra of the frequency response are shown in Figure 8.16. Note that the amplitude and phase spectra are no longer represented by discrete data sets but by continuous functions. Note also that the amplitude spectrum shown in Figure 8.16 is a typical of the *low pass filter*. In other words, the input-output relation defined by the battery and capacitor voltage of the RC circuit can function as a low pass filter. More detailed discussion about *filters* will be presented in Chapters 10 and 12.

PROBLEMS

Consider a periodic signal $x(t)$ whose phase spectrum is shown below.



Problem 8.1 Sketch the phase spectrum of $y(t) = x(-t)$.

Problem 8.2 Sketch the phase spectrum of $y(t) = x(t/2)$.

Problem 8.3 Sketch the phase spectrum of $y(t) = -x(t)$ with an assumption that $X[0] = 0$.

Problem 8.4 Sketch the phase spectrum of $y(t) = 2x(t)$.

Problem 8.5 Sketch the phase spectrum of $y(t) = x(t + 0.25)$.

Problem 8.6 Sketch the phase spectrum of $y(t) = x(t - 2)$.

Problem 8.7 Sketch the phase spectrum of $y(t) = dx(t)/dt$.

Problem 8.8 Sketch the phase spectrum of $y(t) = d^2x(t)/dt^2$.

Problem 8.9 Use the Parseval's theorem and find the average power of the following signal:

$$x(t) = 8\cos(\omega t) - 6\cos(3\omega t) + 4\cos(5\omega t).$$

Problem 8.10 Find the average power of the following signal:

$$x(t) = 3\cos(\omega t) - 2\cos(2\omega t) + 2\cos(5\omega t).$$

Chapter 9

PRINCIPLES OF FOURIER TRANSFORM

Fourier series is a powerful tool for analyzing periodic continuous-time signals. It enables one to see frequency content of a periodic function, to analyze frequency characteristics of a system, and to design filters that remove / strengthen certain frequency components. The application of the Fourier series is, however, limited to periodic signals. In many occasions, we are forced to handle *nonperiodic (or aperiodic) signals*, and Fourier transform is the solution for those who want to analyze the frequency content of a nonperiodic function. Fourier transform is essential in analyzing and processing signals in science and engineering, especially in medical images, computerized axial tomography (CAT), and magnetic resonance imaging (MRI). We thus focus primarily on Fourier transform for the rest of this study.

9.1 WHAT IS FOURIER TRANSFORM?

The fundamental idea that connects Fourier series and Fourier transform can be summarized in the following statement:

> **A nonperiodic signal can be regarded a periodic signal that has an infinitely long fundamental period.**

In other words, one may simplify the basic idea of Fourier transform as follows:

> **Fourier transform is to do Fourier series analysis with an infinitely small fundamental frequency.**

Recall that expressions 7.20 and 7.21 describe Fourier series as

$$x(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t} \quad \text{and} \quad X[m] = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jm\Omega t}\, dt,$$

where Ω is the fundamental frequency of a periodic function $x(t)$, and T_0 the fundamental period. Combining the two expressions, we write

$$x(t) = \sum_{m=-\infty}^{\infty} \left[\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jm\Omega t}\, dt \right] e^{jm\Omega t}.$$

We also recall that $T_0 = 2\pi/\Omega$ and modify the above expression as

$$x(t) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \left[\int_{-\pi/\Omega}^{\pi/\Omega} x(t)\, e^{-jm\Omega t}\, dt \right] e^{jm\Omega t}\, \Omega.$$

Consider now the situation that the fundamental frequency Ω gets infinitely small such that

$$\Omega \longrightarrow d\omega, \quad m\Omega \longrightarrow \omega, \quad \pi/\Omega \longrightarrow \infty, \quad \text{and} \quad \Sigma \longrightarrow \int .$$

With the above approximation, one may describe $x(t)$ as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right] e^{j\omega t}\, d\omega.$$

We then decompose the above expression into the following two expressions:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt, \qquad (9.1)$$

and

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega. \qquad (9.2)$$

Expression 9.1 is the definition of the *Fourier transform* of $x(t)$, and expression 9.2 defines the *inverse Fourier transform* of $X(\omega)$. We also articulate that $x(t)$ and $X(\omega)$ form a Fourier transform pair and symbolically denote the relation as

$$x(t) \quad \Leftrightarrow \quad X(\omega).$$

It is noteworthy that we have already encountered the following Fourier transform pair:

$$h(t) \quad \Leftrightarrow \quad H(\omega),$$

where $h(t)$ and $H(\omega)$ represent the impulse response and frequency response (expression 8.19) of an LTI system, respectively. In other words, the frequency response of an LTI system is, in fact, the Fourier transform of the impulse response of the system. Note also that $X(\omega)$ is, in principle, an extension of the Fourier series coefficient $X[m]$, and Fourier transform is essentially not different from the process of finding Fourier series coefficient. It is thus not surprising that the Fourier transform has a lot of properties similar to those of Fourier series, such as the linearity and Dirichlet conditions.

Not all functions have Fourier transforms. A time function $x(t)$ has the Fourier transform only if it fulfills the *Dirichlet conditions* summarized below (Oppenheim and Willsky, 1997).

1. $x(t)$ should be bounded.
2. $x(t)$ should have a finite number of maxima and minima within any finite interval.
3. $x(t)$ should have a finite number of discontinuities within any finite interval.
4. $x(t)$ should be absolutely integrable, such that

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty.$$

Therefore, absolutely integrable functions that are continuous or discontinuous at only a finite number of locations do have Fourier transforms.

Example 9.1 Find the Fourier transform of the following time function:

$$\begin{aligned} x(t) &= u(t+a)\,u(-t+a) \\ &= u(t+a) - u(t-a), \end{aligned}$$

with $a > 0$.

Solution

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-a}^{a} e^{-j\omega t}\, dt$$

$$= \frac{e^{ja\omega} - e^{-ja\omega}}{j\omega} = \frac{2\sin(a\omega)}{\omega}.$$

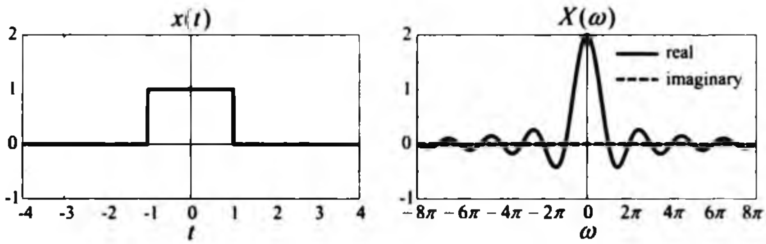


Figure 9.1: A Fourier transform pair. $x(t)$ is an even function of time, and $X(\omega)$ is thus a real function of frequency.

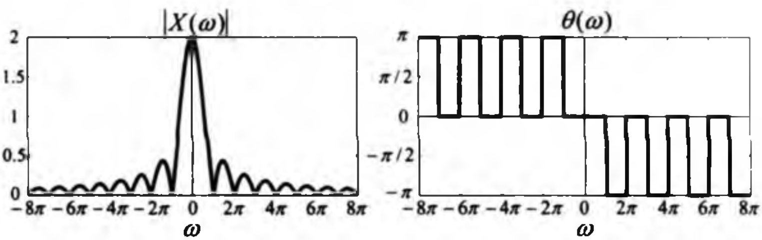


Figure 9.2: Amplitude and phase spectra of the time function $x(t)$ in Figure 9.1. Phase spectrum $\theta(\omega)$ exhibits 0 or $\pm\pi$ values.

Example 9.1 suggests one to write the following Fourier transform pair:

$$x(t) = u(t+1) - u(t-1) \quad \Leftrightarrow \quad X(\omega) = \frac{2\sin\omega}{\omega} = 2\,\mathrm{sinc}(\omega).$$

Figure 9.1 shows the graphs of the above Fourier transform pair. It is obvious that $x(t)$ is an even function of time, and its Fourier transform is thus a real function of frequency. Figure 9.2 demonstrates the amplitude and phase spectra of the time function $x(t)$. It is clear that the amplitude spectrum maintains the even symmetry while the phase spectrum exhibits the odd symmetry.

Example 9.2 Find the Fourier transform of the following time function:

$$x(t) = \sin(\pi t)\, u(t+1)\, u(-t+1).$$

Solution

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-1}^{1} \sin(\pi t)\, e^{-j\omega t}\, dt$$

$$= \int_{-1}^{1} \left[\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right] e^{-j\omega t}\, dt$$

$$= \frac{1}{2j}\left[\int_{-1}^{1} e^{j(\pi-\omega)t}\, dt - \int_{-1}^{1} e^{-j(\pi+\omega)t}\, dt\right]$$

$$= \frac{e^{j(\pi-\omega)} - e^{-j(\pi-\omega)}}{2j^2(\pi-\omega)} - \frac{e^{j(\pi+\omega)} - e^{-j(\pi+\omega)}}{2j^2(\pi+\omega)}$$

$$= \frac{\sin(\pi-\omega)}{j(\pi-\omega)} - \frac{\sin(\pi+\omega)}{j(\pi+\omega)}$$

$$= \frac{j\sin(\pi+\omega)}{\pi+\omega} - \frac{j\sin(\pi-\omega)}{\pi-\omega}.$$

Figure 9.3 shows the Fourier transform pair derived in Example 9.2. It is obvious that $x(t)$ is an odd function of time and its Fourier transform is thus an imaginary function of frequency. Figure 9.4 demonstrates the amplitude and phase spectra of the time function $x(t)$. It is clear that the amplitude and phase spectra still maintains the even and odd symmetry, respectively.

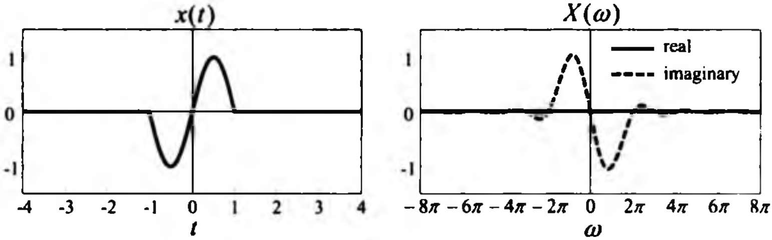


Figure 9.3: A Fourier transform pair. $x(t)$ is an odd function of time, and $X(\omega)$ is thus an imaginary function of frequency.

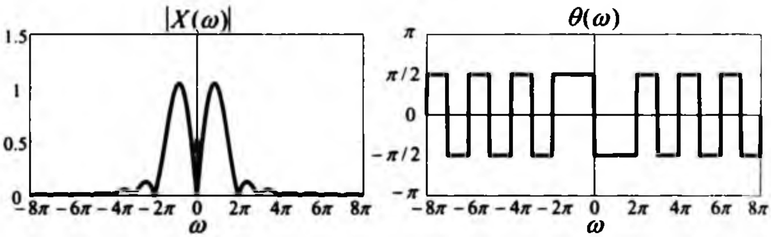


Figure 9.4: Amplitude and phase spectra of the time function $x(t)$ in Figure 9.3. Phase spectrum $\theta(\omega)$ exhibits 0 or $\pm\pi/2$ values.

Example 9.3 Find the Fourier transform of the following time function:

$$x(t) = e^{-at}\, u(t),$$

with $a > 0$.

Solution

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\ &= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt \\ &= \frac{1}{a + j\omega}. \end{aligned}$$

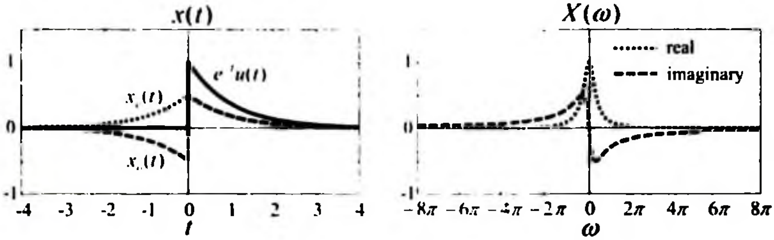


Figure 9.5: A Fourier transform pair

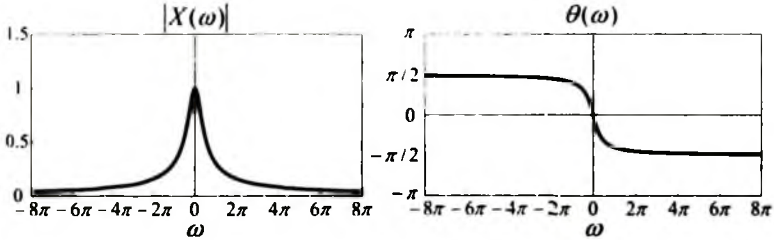


Figure 9.6: Amplitude and phase spectra of the time function $x(t)$ in Figure 9.5

Example 9.3 enables one to write the following Fourier transform pair:

$$x(t) = e^{-t}\,u(t) \quad \Leftrightarrow \quad X(\omega) = \frac{1}{1 + j\omega}.$$

Figure 9.5 shows the graphs of the above Fourier transform pair. It is obvious that $x(t)$ is neither an even nor an odd function of time, and its Fourier transform is thus a complex function of frequency. It is also noteworthy that the real and imaginary parts of $X(\omega)$, in fact, correspond to the Fourier transforms of the even and odd parts of $x(t)$, respectively. Figure 9.6 demonstrates the amplitude and phase spectra of the time function $x(t)$. It is clear that as expected, the amplitude spectrum exhibits the even symmetry while the phase spectrum maintains the odd symmetry.

9.2 PROPERTIES OF FOURIER TRANSFORM

Linearity

We assume that the Fourier transforms of $x_1(t)$ and $x_2(t)$ exist as follows:

$$X_1(\omega) = \int_{-\infty}^{\infty} x_1(t)\,e^{-j\omega t}\,dt \quad \text{and} \quad X_2(\omega) = \int_{-\infty}^{\infty} x_2(t)\,e^{-j\omega t}\,dt.$$

We also assume that $y(t)$ is a linear combination of $x_1(t)$ and $x_2(t)$ such that

$$y(t) = ax_1(t) + bx_2(t). \tag{9.3}$$

The Fourier transform of $y(t)$ then becomes

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} [ax_1(t) + bx_2(t)]\, e^{-j\omega t}\, dt \\ &= a\int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt + b\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt, \end{aligned}$$

and we simplify the above expression as

$$Y(\omega) = aX_1(\omega) + bX_2(\omega). \tag{9.4}$$

Expressions 9.3 and 9.4 demonstrate the linearity of the Fourier transform. And the linearity can be extended to a linear combination of an arbitrary number of functions.

Time Reversal

We assume that the Fourier transform of $x(t)$ exists as follow:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

We also assume that $y(t)$ is the *time reversal* of $x(t)$ such that

$$y(t) = x(-t). \tag{9.5}$$

The Fourier transform of $y(t)$ then becomes

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega t}\, dt.$$

Substituting $\tau = -t$ and $d\tau = -dt$, we write

$$Y(\omega) = -\int_{\infty}^{-\infty} x(\tau)\, e^{j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} x(\tau)\, e^{j\omega\tau}\, d\tau$$

and finally express the Fourier transform of $y(t)$ as

$$Y(\omega) = X(-\omega). \tag{9.6}$$

Time Scaling

We assume that the Fourier transform of $x(t)$ exists as follow:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

We also assume that $y(t)$ is the *time scaling* of $x(t)$ such that

$$y(t) = x(at) \tag{9.7}$$

with $a > 0$. The Fourier transform of $y(t)$ then becomes

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt.$$

Substituting $\tau = at$ and $d\tau = adt$, we write

$$Y(\omega) = \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau/a}\, d\tau$$

and finally express the Fourier transform of $y(t)$ as

$$Y(\omega) = \frac{1}{a}\, X(\omega/a). \tag{9.8}$$

Time Shifting

We assume that the Fourier transform of $x(t)$ exists as follow:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

We also assume that $y(t)$ is the *time shifting* of $x(t)$ such that

$$y(t) = x(t - t_0) \tag{9.9}$$

with a constant t_0. The Fourier transform of $y(t)$ then becomes

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(t - t_0)\, e^{-j\omega t}\, dt.$$

Substituting $\tau = t - t_0$ and $d\tau = dt$, we write

$$Y(\omega) = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega(\tau+t_0)}\, d\tau = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

and finally express the Fourier transform of $y(t)$ as

$$Y(\omega) = e^{-j\omega t_0}\, X(\omega). \tag{9.10}$$

Frequency Shifting

We assume that the Fourier transform of $x(t)$ exists as follows:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

We also assume that $y(t)$ is expressed as

$$y(t) = x(t)\, e^{j\omega_0 t} \tag{9.11}$$

with a constant ω_0. The Fourier transform of $y(t)$ then becomes

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{j\omega_0 t}\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt, \end{aligned}$$

and we finally express the Fourier transform of $y(t)$ as

$$Y(\omega) = X(\omega - \omega_0). \tag{9.12}$$

Note that expressions 9.11 and 9.12 demonstrate the *frequency shifting* relation between $x(t)$ and $y(t)$.

Example 9.4 Use the following Fourier transform pair:

$$x(t) = e^{-t}\, u(t) \quad \Leftrightarrow \quad X(\omega) = \frac{1}{1 + j\omega},$$

and derive Fourier transforms of the following expressions:

$$\begin{aligned} y_1(t) &= e^{t}\, u(-t), & y_2(t) &= e^{-|t|}, \\ y_3(t) &= e^{-2t}\, u(t), & y_4(t) &= e^{2t}\, u(-t), \\ y_5(t) &= e^{-t+2}\, u(t-2), & y_6(t) &= e^{(2j-1)t}\, u(t). \end{aligned}$$

Solution

$y_1(t) = x(-t)$,

$$Y_1(\omega) = X(-\omega) = \frac{1}{1 - j\omega}.$$

$y_2(t) = e^{-t}\, u(t) + e^{t}\, u(-t) = x(t) + x(-t).$

$$Y_2(\omega) = X(\omega) + X(-\omega) = \frac{1}{1+j\omega} + \frac{1}{1-j\omega} = \frac{2}{1+\omega^2}.$$

$y_3(t) = x(2t),$

$$Y_3(\omega) = \frac{1}{2}\, X(\omega/2) = \frac{1}{2}\,\frac{1}{1+j\omega/2} = \frac{1}{2+j\omega}.$$

$y_4(t) = x(-2t) = y_3(-t),$

$$Y_4(\omega) = Y_3(-\omega) = \frac{1}{2-j\omega}.$$

$y_5(t) = x(t-2),$

$$Y_5(\omega) = e^{-2j\omega}\, X(\omega) = \frac{e^{-2j\omega}}{1+j\omega}.$$

$y_6(t) = x(t)\, e^{2jt},$

$$Y_6(\omega) = X(\omega - 2) = \frac{1}{1+j(\omega-2)}.$$

Time Differentiation

We assume that the inverse Fourier transform of $X(\omega)$ exists as follows:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega.$$

We also assume that $y(t)$ is the *time differentiation* of $x(t)$ such that

$$y(t) = \frac{dx(t)}{dt}. \tag{9.13}$$

Note that expression 9.13 becomes

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(\omega)\, e^{j\omega t}\, d\omega,$$

which implies that $y(t)$ is the inverse Fourier transform of $j\omega X(\omega)$. In other words, $j\omega X(\omega)$ is the Fourier transform of $y(t)$ such that

$$Y(\omega) = j\omega X(\omega). \tag{9.14}$$

Duality

We assume a Fourier transform pair that is described as follow:

$$x(t) \quad \Leftrightarrow \quad X(\omega). \tag{9.15}$$

Time function $x(t)$ is then expressed as the inverse Fourier transform of $X(\omega)$ such that

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega.$$

Substituting t with $-t$, we rewrite the above expression as

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega.$$

And interchanging the two variables t and ω yields

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt.$$

The above expression means that the "frequency" function $2\pi x(-\omega)$ is the Fourier transform of the "time" function $X(t)$. One can therefore write the following Fourier transform pair:

$$X(t) \quad \Leftrightarrow \quad 2\pi x(-\omega). \tag{9.16}$$

Expressions 9.15 and 9.16 manifest that Fourier transform pairs are "almost symmetric", and we call the semi-symmetric nature the *duality* of Fourier transform. An example of the duality will be soon presented in terms of the Fourier transform of the unit impulse function.

9.3 FOURIER TRANSFORM OF SPECIAL FUNCTIONS

9.3.1 Fourier Transform of Unit Impulse Function

Fourier transform is a versatile tool that can even enable one to discuss the frequency contents of the *singularity functions*. The Fourier transforms of the *unit impulse functions* are of our special interest. Consider the Fourier transform of $x(t) = \delta(t - t_0)$ as follows:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt.$$

It is evident that the *sifting property* of the impulse function (expression 1.6) simplifies the above expression as $X(\omega) = e^{-j\omega t_0}$. In other words, one may write the following Fourier transform pair:

$$\delta(t - t_0) \quad \Leftrightarrow \quad e^{-j\omega t_0} \tag{9.17}$$

and express $\delta(t - t_0)$ as the inverse Fourier transform of $e^{-j\omega t_0}$ such that

$$\delta(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(t - t_0)}\, d\omega. \tag{9.18}$$

It is noteworthy that expression 9.18 is, in fact, the most rigorous way of defining the impulse functions.

Consider now a special case of expression 9.17. Assigning $t_0 = 0$ yields

$$\delta(t) \quad \Leftrightarrow \quad 1, \tag{9.19}$$

which implies that the Fourier transform of $\delta(t)$ is just 1. It may look trivial, but expression 9.19 has an important physical implication that the impulsive excitation at $t = 0$ contains "every" frequency components and each different frequency component is of equal significance (constant amplitude spectrum). Consider also Fourier transforming $x(t) = 1$ as follows:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-j\omega t}\, dt.$$

Note that the above integration is impossible to evaluate without introducing impulse function. We thus recall the duality of the Fourier transform (expressions 9.15 and 9.16) and write the following Fourier transform pair:

$$1 \quad \Leftrightarrow \quad 2\pi\delta(-\omega) = 2\pi\delta(\omega). \tag{9.20}$$

The duality of expressions 9.19 and 9.20 are summarized in Figure 9.7. The duality of the Fourier transform can be also applied to expression 9.17 such that

$$e^{-j\omega_0 t} \quad \Leftrightarrow \quad 2\pi\delta(-\omega - \omega_0) = 2\pi\delta(\omega + \omega_0). \tag{9.21}$$

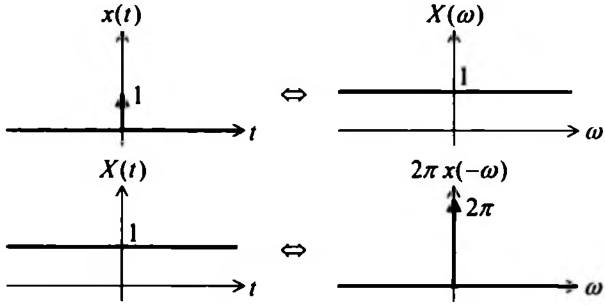
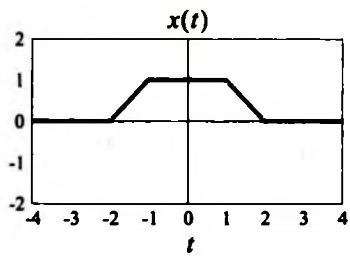


Figure 9.7: An example of the duality of the Fourier transform

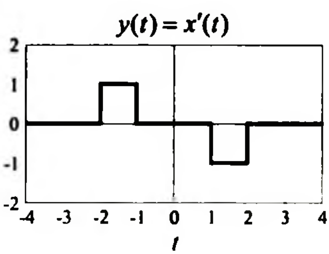
Note that applying the frequency shifting property to expression 9.20 also yields expression 9.21.

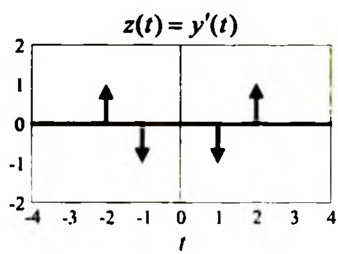
Example 9.5 Use the time differentiation property (expressions 9.13 and 9.14) and derive the Fourier transform of $x(t)$ shown below.



Solution

We sketch time derivatives of $x(t)$ and derive $X(\omega)$ as follows.





$$z(t) = \delta(t+2) - \delta(t+1) - \delta(t-1) + \delta(t-2),$$

$$Z(\omega) = e^{2j\omega} - e^{j\omega} - e^{-j\omega} + e^{-2j\omega}$$

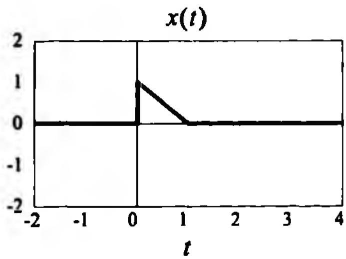
$$= 2[\cos(2\omega) - \cos(\omega)],$$

$$z(t) = \frac{d^2x(t)}{dt^2},$$

$$Z(\omega) = (j\omega)^2 X(\omega),$$

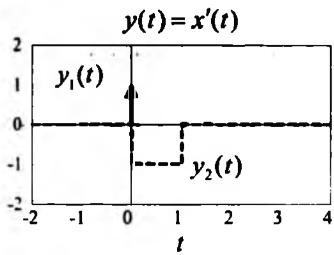
$$X(\omega) = -\frac{Z(\omega)}{\omega^2} = 2\frac{\cos(\omega) - \cos(2\omega)}{\omega^2}.$$

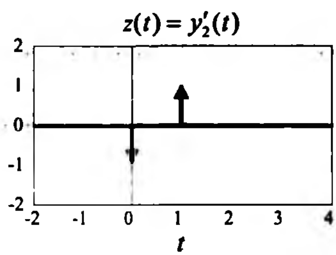
Example 9.6 Use the time differentiation property (expressions 9.13 and 9.14) and derive the Fourier transform of $x(t)$ shown below.



Solution

We sketch time derivatives of $x(t)$ and derive $X(\omega)$ as follows.





$$
\begin{aligned}
y(t) &= x'(t) = y_1(t) + y_2(t) \\
z(t) &= y_2'(t) = -\delta(t) + \delta(t-1), \\
Z(\omega) &= -1 + e^{-j\omega} = (j\omega)\,Y_2(\omega), \\
Y_2(\omega) &= \frac{-1 + e^{-j\omega}}{j\omega}, \\
y(t) &= \delta(t) + y_2(t), \\
Y(\omega) &= 1 + Y_2(\omega) = 1 + \frac{-1 + e^{-j\omega}}{j\omega} = (j\omega)\,X(\omega), \\
X(\omega) &= \frac{1}{j\omega} + \frac{1 - e^{-j\omega}}{\omega^2}.
\end{aligned}
$$

Examples 9.5 and 9.6 show that combined with the Fourier transform of the impulse function, time differentiation property can ease deriving Fourier transform of complex time functions. Note, however, that we should be cautious to follow the aforementioned process if the time function $x(t)$ contains a DC component. As long as $x(t)$ has no DC component, it is safe to follow the process. The unit step function is a good example of a time function that contains a DC component.

9.3.2 Fourier Transform of Unit Step Function

The *unit step function* $u(t)$ has a DC component, and one can quickly verify the existence of a DC component by evaluating the even part of the function. The even part of the unit step function $u_e(t)$ is

$$u_e(t) = \frac{u(t) + u(-t)}{2} = \frac{1}{2},$$

and we thus express the unit step function as

$$u(t) = \frac{1}{2} + u_o(t),$$

where $u_o(t)$ is the odd part of the unit step function. Figure 9.8 shows the even and odd parts of the unit step function, and it is obvious that the unit

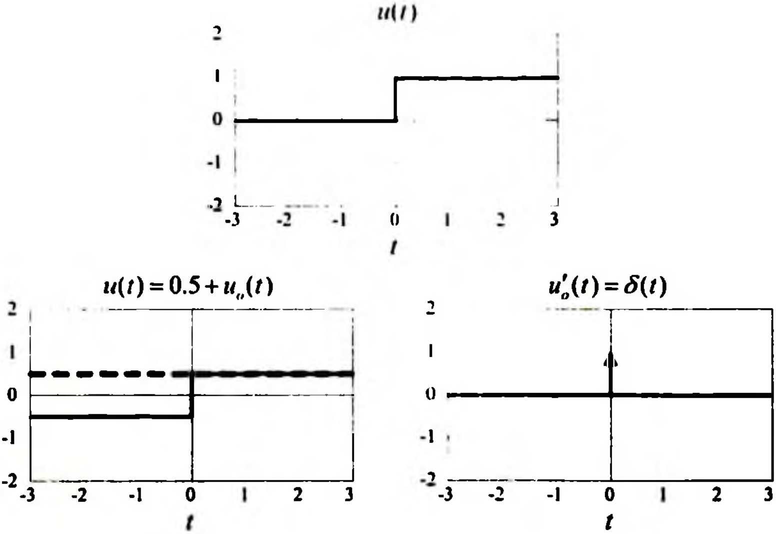


Figure 9.8: Unit step function and its geometric analysis

step function contains a DC component. The Fourier transform of $u_o(t)$ is derived as follows:

$$u_o'(t) = \delta(t), \quad (j\omega)\, U_o(\omega) = 1, \quad \text{and} \quad U_o(\omega) = \frac{1}{j\omega}.$$

We also recall that

$$\frac{1}{2} \Leftrightarrow \pi\, \delta(\omega),$$

and write the following Fourier transform pair:

$$u(t) \quad \Leftrightarrow \quad \pi\, \delta(\omega) + \frac{1}{j\omega}. \tag{9.22}$$

9.3.3 Fourier Transform of Sinusoidal Functions

The Fourier transform has originated from an effort to expand the domain of Fourier series from the periodic functions to nonperiodic ones. The application of Fourier transform is, however, not limited to nonperiodic functions. One can express the Fourier transform of, for example, *sinusoidal functions*. We know cosine functions are expressed as

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2},$$

and expression 9.21 allows one to write the Fourier transform of the above expression as

$$\frac{2\pi\delta(\omega-\omega_0)+2\pi\delta(\omega+\omega_0)}{2}.$$

We can thus derive the following Fourier transform pair:

$$\cos(\omega_0 t) \quad \Leftrightarrow \quad \pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]. \qquad (9.23)$$

Similarly, one can express sine functions as

$$\sin(\omega_0 t) = \frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j},$$

utilize expression 9.21 to get the Fourier transform of sine functions as

$$\frac{2\pi\delta(\omega-\omega_0)-2\pi\delta(\omega+\omega_0)}{2j},$$

and finally write the following Fourier transform pair:

$$\sin(\omega_0 t) \quad \Leftrightarrow \quad j\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]. \qquad (9.24)$$

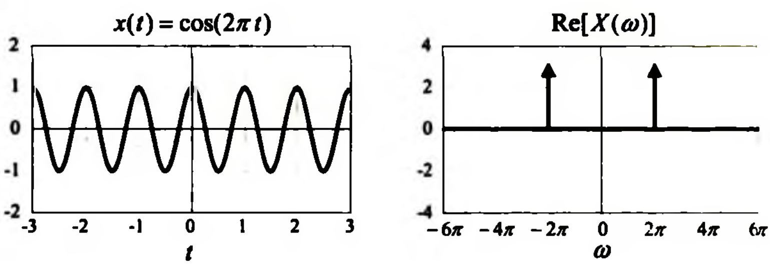


Figure 9.9: A cosine function and its Fourier transform

Figures 9.9 and 9.10 show examples of Fourier transforming cosine and sine functions. It is evident that the Fourier transforms of cosine functions have only real parts and those real parts exhibit even symmetry. On the other hand, Fourier transforming sine functions yield only imaginary parts and those imaginary parts maintain odd symmetry.

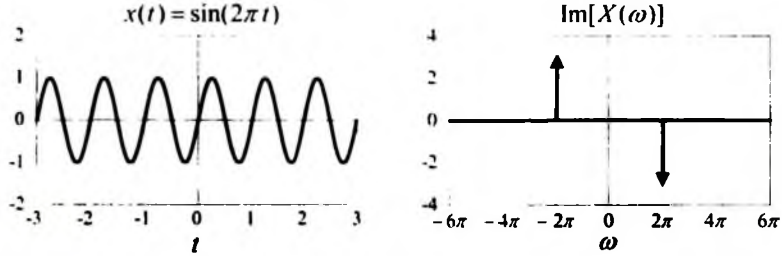


Figure 9.10: A sine function and it Fourier transform

9.4 PARSEVAL'S THEOREM

We have argued in Chapter 2 that the total energy E of a continuous-time signal is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt. \tag{9.25}$$

One may use the following expressions:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \quad \text{and} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

and describe the total energy as

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) x^*(t)\, dt \\ &= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t}\, d\omega \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) X(\omega)\, d\omega. \end{aligned}$$

In other words, the total energy E can be expressed as

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega. \tag{9.26}$$

The above expression is called the *Parseval's theorem*. Parseval's theorem states the relationship between energy in the time and frequency domains, and the theorem plays a significant role in communications and signal processing.

9.5 SUMMARY

Several important properties of the Fourier transform are summarized in Table 9.1. Time convolution and frequency convolution properties will be discussed in Chapter 10. Fourier transform pairs of several common functions are also summarized in Table 9.2.

Table 9.1: Properties of Fourier transform

Property	$x(t)$	$X(\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Time Reversal	$x(-t)$	$X(-\omega)$
Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X(\omega/a)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$
Time Differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(\omega) X_2(\omega)$
Frequency convolution	$x_1(t) x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

Table 9.2: Fourier transform pairs

$x(t)$	$X(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$e^{-j\omega_0 t}$	$2\pi\delta(\omega + \omega_0)$
$u(t)$	$\pi\,\delta(\omega) + \frac{1}{j\omega}$
$u(t + a) - u(t - a)$	$\frac{2\sin(a\omega)}{\omega}$
$e^{-at}\,u(t)$	$\frac{1}{a + j\omega}$
$e^{at}\,u(-t)$	$\frac{1}{a - j\omega}$
$e^{-a\|t\|}$	$\frac{2a}{a^2 + \omega^2}$
$\|t\|e^{-at}\,u(t)$	$\frac{j4a\omega}{a^2 + \omega^2}$
$t^n\,e^{-at}\,u(t)$	$\frac{n!}{(a + j\omega)^{n+1}}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
$e^{-at}\cos(\omega_0 t)\,u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$
$e^{-at}\sin(\omega_0 t)\,u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$

PROBLEMS

Problem 9.1 Fourier transform of a time signal $x(t)$ is given as

$$X(\omega) = \frac{9}{9+\omega^2}.$$

Which of the following is correct? Choose one.

a. $y(t) = x(t)e^{-3jt} \quad \Leftrightarrow \quad Y(\omega) = \dfrac{9}{9+(\omega+3)^2}$

b. $y(t) = x(t+3) \quad \Leftrightarrow \quad Y(\omega) = \dfrac{9\,e^{-3j\omega}}{9+\omega^2}$

c. $y(t) = x(t/3) \quad \Leftrightarrow \quad Y(\omega) = \dfrac{3}{9+(\omega/3)^2}$

d. $y(t) = x(-3t) \quad \Leftrightarrow \quad Y(\omega) = \dfrac{-3}{9+(\omega/3)^2}$

Problem 9.2 Fourier transform of a time signal $x(t)$ is given as

$$X(\omega) = \frac{9}{9+\omega^2}.$$

Which of the following is correct? Choose one.

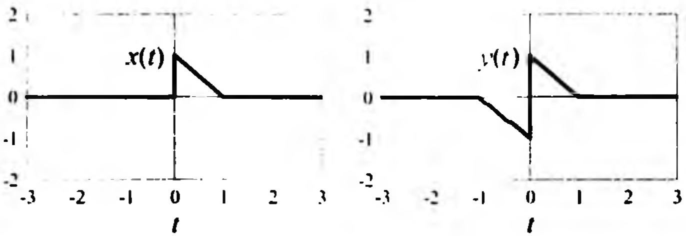
a. $y(t) = x(t)e^{3jt} \quad \Leftrightarrow \quad Y(\omega) = \dfrac{9}{9+(\omega+3)^2}$

b. $y(t) = x(t-3) \quad \Leftrightarrow \quad Y(\omega) = \dfrac{9\,e^{3j\omega}}{9+\omega^2}$

c. $y(t) = x(t/3) \quad \Leftrightarrow \quad Y(\omega) = \dfrac{3}{9+(3\omega)^2}$

d. $y(t) = x(-3t) \quad \Leftrightarrow \quad Y(\omega) = \dfrac{3}{9+(\omega/3)^2}$

Problem 9.3 Consider time signals shown below.

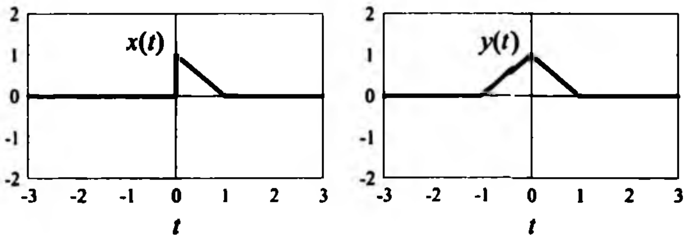


The Fourier transform of $x(t)$ is given as

$$x(t) \quad \Leftrightarrow \quad X(\omega) = \frac{1}{j\omega} + \frac{1 - e^{-j\omega}}{\omega^2}.$$

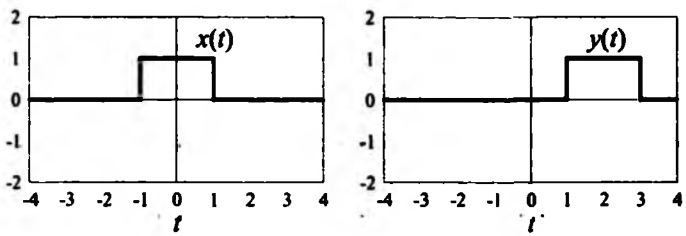
Derive the Fourier transform of $y(t)$.

Problem 9.4 Consider time signals shown below.



Use the Fourier transform pair given in Problem 9.3 and derive the Fourier transform of $y(t)$.

Problem 9.5 Consider time signals shown below.



The Fourier transform of $x(t)$ is given as

$$x(t) \quad \Leftrightarrow \quad X(\omega) = \frac{2 \sin \omega}{\omega}.$$

Derive the Fourier transform of $y(t)$.

Problem 9.6 Use the following property:

$$\frac{dy(t)}{dt} \quad \Leftrightarrow \quad j\omega Y(\omega),$$

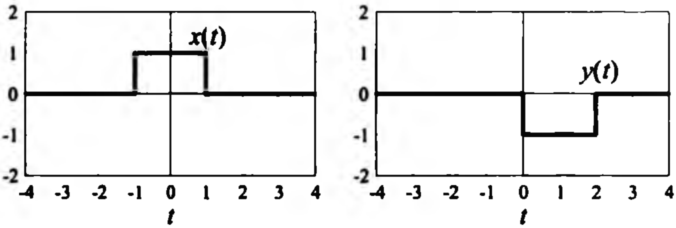
and derive the Fourier transform of $y(t)$ shown in Problem 9.5.

Problem 9.7 Use the following definition:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt,$$

and derive the Fourier transform of $y(t)$ shown in Problem 9.5.

Problem 9.8 Consider time signals shown below.



The Fourier transform of $x(t)$ is given as

$$x(t) \quad \Leftrightarrow \quad X(\omega) = \frac{2\sin\omega}{\omega}.$$

Derive the Fourier transform of $y(t)$.

Problem 9.9 Use the following property:

$$\frac{dy(t)}{dt} \quad \Leftrightarrow \quad j\omega Y(\omega),$$

and derive the Fourier transform of $y(t)$ shown in Problem 9.8.

Problem 9.10 Use the following definition:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt,$$

and derive the Fourier transform of $y(t)$ shown in Problem 9.8.

Chapter 10

FOURIER TRANSFORM AND LTI SYSTEMS

10.1 CONVOLUTION PROPERTIES

Assume that $y(t)$ is the convolution of two continuous-time functions $x_1(t)$ and $x_2(t)$:

$$y(t) = x_1(t) * x_2(t). \tag{10.1}$$

Assume also that $x_1(t)$ and $x_2(t)$ have Fourier transforms. We then express the Fourier transform of $y(t)$ as

$$\begin{aligned}
Y(\omega) &= \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x_1(t) * x_2(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t} dt \right] d\tau.
\end{aligned}$$

Substituting $\zeta = t - \tau,\ t = \zeta + \tau,$ and $dt = d\zeta$, we rewrite $Y(\omega)$ as

$$\begin{aligned}
Y(\omega) &= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(\zeta)\, e^{-j\omega(\zeta+\tau)} d\zeta \right] d\tau \\
&= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} x_2(\zeta)\, e^{-j\omega\zeta} d\zeta,
\end{aligned}$$

and finally express the Fourier transform of $y(t)$ as

$$Y(\omega) = X_1(\omega)\, X_2(\omega), \tag{10.2}$$

The above expression means that Fourier transforming the convolution of two time functions is to take the product of their each corresponding Fourier transforms. Expressions 10.1 and 10.2 are called the *time convolution* property of Fourier transform.

Motivated by the duality of Fourier transform, one may also consider Fourier transforming the product of two time functions such that

$$y(t) = x_1(t)\, x_2(t), \tag{10.3}$$

and

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-j\omega t}\, dt.$$

Denoting $x_1(t)$ as the inverse Fourier transform of $X_1(\eta)$ yields

$$Y(\omega) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\eta)\, e^{j\eta t} d\eta \right] x_2(t)\, e^{-j\omega t}\, dt.$$

Note that while representing the inverse Fourier transform, another frequency variable η should be used to make distinction from the frequency variable ω, because η and ω may vary independently from each other. One can then rewrite $Y(\omega)$ as

$$\begin{aligned} Y(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\eta) \left[\int_{-\infty}^{\infty} x_2(t)\, e^{-j(\omega-\eta)t} dt \right] d\eta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\eta) X_2(\omega - \eta)\, d\eta, \end{aligned}$$

and finally express the Fourier transform of $y(t)$ as

$$Y(\omega) = \frac{1}{2\pi} X_1(\omega) * X_2(\omega). \tag{10.4}$$

Expressions 10.3 and 10.4 represent the *frequency convolution* property of the Fourier transform.

In many occasions, convolution properties of Fourier transform allows one to avoid handling tough convolution problems and to find an alternative route of acquiring solutions. And, as a result of that, Fourier transform plays a significant role for analyzing LTI systems.

10.2 FREQUENCY RESPONSE OF LTI SYSTEMS

We have studied in Chapter 5 that for an LTI system, the input signal $x(t)$ and output signal $y(t)$ are associated as

$$y(t) = x(t) * h(t),$$

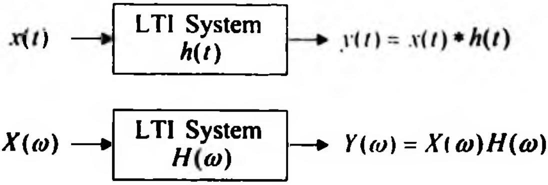


Figure 10.1: Time domain and frequency domain representations of an LTI system

where $h(t)$ denotes the impulse response of the system. Applying the time convolution property of Fourier transform to the above expression results in

$$Y(\omega) = X(\omega)\,H(\omega), \tag{10.5}$$

where $H(\omega)$ is the *frequency response* of the LTI system. The frequency response is derived by the Fourier transform of the impulse response such that

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\,e^{-j\omega t}\,dt. \tag{10.6}$$

Representing characteristics of a system via frequency response eases associating input and output signals of complicated *interconnected systems*. Consider, for example, interconnected systems depicted in Figure 10.2. Input-output relations of the interconnected systems are expressed as

$$Y_1(\omega) = X(\omega)\,H_1(\omega)\,H_2(\omega),$$

$$Y_2(\omega) = X(\omega)\,[H_1(\omega) + H_2(\omega)],$$

$$Y_3(\omega) = X(\omega)\,H_1(\omega)\,[H_2(\omega) + H_3(\omega)],$$

$$Y_4(\omega) = X(\omega)\,[H_1(\omega)\,H_2(\omega) + H_3(\omega)].$$

It is obvious that representing input-output relation is simpler without convolution.

Example 10.1 Use the following definition of the frequency response:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)},$$

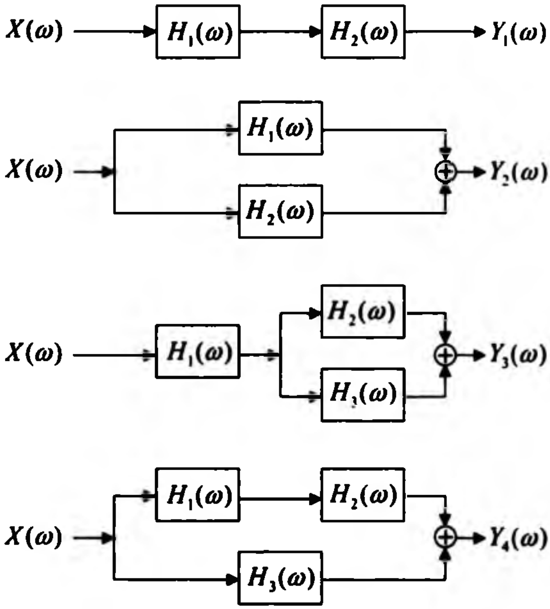
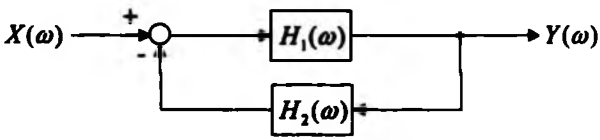
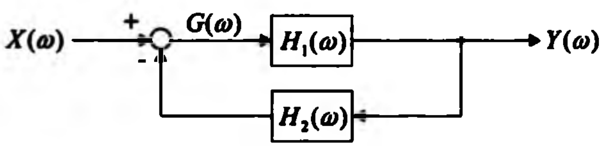


Figure 10.2: Interconnected systems

and derive the frequency response of the feedback system shown below.



Solution



Denoting the input to system-1 as $G(\omega)$, we describe $G(\omega)$ as

$$G(\omega) = X(\omega) - Y(\omega)\, H_2(\omega).$$

We also associate $G(\omega)$ with the output from the system as

$$Y(\omega) = G(\omega)\, H_1(\omega).$$

Combining the above two expressions yields

$$Y(\omega) = X(\omega)\, H_1(\omega) - Y(\omega)\, H_2(\omega)\, H_1(\omega),$$

$$Y(\omega)\, [1 + H_2(\omega)\, H_1(\omega)] = X(\omega)\, H_1(\omega).$$

And we express the frequency response as

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H_1(\omega)}{1 + H_2(\omega)\, H_1(\omega)}.$$

Note that Example 10.1 exemplifies negative feedback systems. Positive feedback systems may cause system instability and are thus less common than negative feedback systems.

Example 10.2 Derive the frequency responses of LTI systems the following differential equations represent:

$$\frac{1}{5}\frac{dy_1(t)}{dt} + y_1(t) = x_1(t),$$

$$\frac{dy_2(t)}{dt} + 5y_2(t) = \frac{dx_2(t)}{dt} + x_2(t),$$

$$\frac{d^2y_3(t)}{dt^2} + 2\frac{dy_3(t)}{dt} + y_3(t) = x_3(t).$$

Solution

Fourier transforming the differential equations yields

$$j\frac{\omega}{5} Y_1(\omega) + Y_1(\omega) = X_1(\omega),$$

$$j\omega\, Y_2(\omega) + 5Y_2(\omega) = j\omega\, X_2(\omega) + X_2(\omega),$$

$$(j\omega)^2\, Y_3(\omega) + j2\omega\, Y_3(\omega) + Y_3(\omega) = X_3(\omega).$$

> The frequency responses are thus expressed as
>
> $$H_1(\omega) = \frac{Y_1(\omega)}{X_1(\omega)} = \frac{5}{5 + j\omega},$$
> $$H_2(\omega) = \frac{Y_2(\omega)}{X_2(\omega)} = \frac{1 + j\omega}{5 + j\omega},$$
> $$H_3(\omega) = \frac{Y_3(\omega)}{X_3(\omega)} = \frac{1}{(1 + j\omega)^2}.$$

Note that frequency response $H_1(\omega)$ in Example 10.2 has been already derived in Example 8.5. The derivation was based on the impulse response of an RC circuit, which we had discussed in Chapter 5. Compared to the hardship one has to endure while deriving the frequency response via time domain study, it is surprisingly straightforward to derive the same frequency response in the frequency domain. This favorable aspect of the frequency domain analysis strongly motivates one to study Fourier transform. We should note, however, that to fully enjoy the advantage of the frequency domain analysis, it is necessary to verify one more remaining hurdle. And the hurdle is the inverse Fourier transform.

10.3 INVERSE FOURIER TRANSFORM

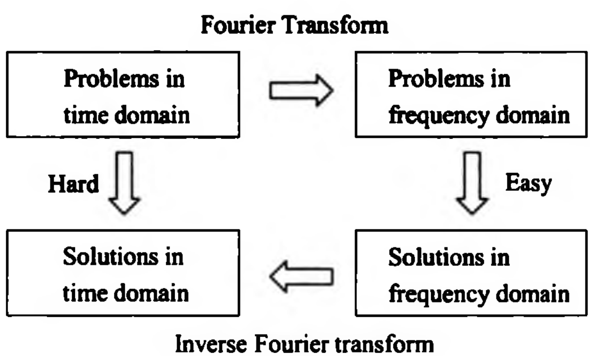


Figure 10.3: Forward and inverse Fourier transform

Figure 10.3 illustrates a workflow one frequently encounters. Having a challenging problem in time domain, one diverts into the frequency domain via the Fourier transform, solve the problem in the frequency domain, and finally return to the time domain via the ***inverse Fourier transform***. In general, we use expression 9.2 to find the inverse Fourier transform of $X(\omega)$. Quite often, however, $X(\omega)$ is in the form of a rational function. And, in such a situation, we perform a partial fraction expansion of $X(\omega)$ and find the corresponding time domain signals $x(t)$ by referring to Fourier transform pairs that have already been identified (Table 9.2). Furthermore, we frequently need to utilize properties of Fourier transform to ease the process of finding inverse Fourier transforms (Table 9.1).

Example 10.3 Find the inverse Fourier transform of

$$H(\omega) = \frac{5}{5 + j\omega}.$$

Solution

$$h(t) = 5e^{-5t}u(t).$$

Example 10.4 Find the inverse Fourier transform of

$$H(\omega) = \frac{1 + j\omega}{5 + j\omega}.$$

Solution

$$H(\omega) = \frac{5 + j\omega - 4}{5 + j\omega} = 1 - \frac{4}{5 + j\omega},$$

$$h(t) = \delta(t) - 4e^{-5t}\,u(t).$$

Example 10.5 Find the inverse Fourier transform of

$$H(\omega) = \frac{1}{(1 + j\omega)^2}.$$

Solution

$$h(t) = t\, e^{-t}\, u(t).$$

Example 10.6 Find the inverse Fourier transform of

$$H(\omega) = \frac{j2\omega}{1 + j\omega}.$$

Solution

$$H(\omega) = \frac{2 + j2\omega - 2}{1 + j\omega} = 2 - \frac{2}{1 + j\omega},$$

$$h(t) = 2\delta(t) - 2e^{-t}\, u(t).$$

Another approach is to use the time differentiation property as follows:

$$H(\omega) = j\omega\, G(\omega), \qquad G(\omega) = \frac{2}{1 + j\omega},$$

$$g(t) = 2e^{-t}\, u(t),$$

$$\begin{aligned} h(t) = \frac{dg(t)}{dt} &= -2e^{-t}\, u(t) + 2e^{-t}\, \frac{du(t)}{dt} \\ &= -2e^{-t}\, u(t) + 2e^{-t}\, \delta(t) \\ &= -2e^{-t}\, u(t) + 2\delta(t). \end{aligned}$$

Example 10.7 Find the inverse Fourier transform of

$$H(\omega) = \frac{e^{-j2\omega}}{1 + j\omega}.$$

Solution

$$H(\omega) = e^{-j2\omega}G(\omega), \qquad G(\omega) = \frac{1}{1 + j\omega},$$

$$g(t) = e^{-t}u(t), \qquad h(t) = g(t-2) = e^{-(t-2)}u(t-2).$$

Example 10.8 Find the inverse Fourier transform of

$$H(\omega) = \frac{-1}{1 + j\omega + 2j}.$$

Solution

$$H(\omega) = \frac{-1}{1 + j(\omega + 2)} = G(\omega + 2), \qquad G(\omega) = \frac{-1}{1 + j\omega},$$

$$g(t) = -e^{-t}u(t), \qquad h(t) = e^{-2jt}g(t) = -e^{-(1+2j)t}u(t).$$

Example 10.9 Find the inverse Fourier transform of

$$Y(\omega) = \frac{j10\omega}{(2 - j\omega)(3 + j\omega)}.$$

Solution

$$Y(\omega) = \frac{j10\omega}{(2 - j\omega)(3 + j\omega)} = \frac{A}{2 - j\omega} + \frac{B}{3 + j\omega},$$

$$\begin{cases} 3A + 2B = 0, \\ A - B = 10, \end{cases} \quad \rightarrow \quad \begin{cases} A = 4, \\ B = -6, \end{cases}$$

$$Y(\omega) = \frac{4}{2 - j\omega} - \frac{6}{3 + j\omega},$$

$$y(t) = 4e^{2t}u(-t) - 6e^{-3t}u(t).$$

Example 10.10 Find the inverse Fourier transform of

$$Y(\omega) = \frac{5}{6 + j5\omega - \omega^2}.$$

Solution

$$Y(\omega) = \frac{5}{6 + j5\omega + (j\omega)^2} = \frac{5}{(2 + j\omega)(3 + j\omega)} = \frac{A}{2 + j\omega} + \frac{B}{3 + j\omega},$$

$$\begin{cases} 3A + 2B = 5, \\ A + B = 0, \end{cases} \quad \rightarrow \quad \begin{cases} A = 5, \\ B = -5, \end{cases}$$

$$Y(\omega) = \frac{5}{2 + j\omega} - \frac{5}{3 + j\omega},$$

$$y(t) = 5e^{-2t}u(t) - 5e^{-3t}u(t).$$

10.4 APPLICATIONS

Fourier transform is extensively used in a variety of fields that include optics, spectroscopy, acoustics, geophysics, and electrical engineering. In electrical engineering, it is applied in circuit analysis, communications systems, signal processing, and so on. We discuss three application areas: circuit analysis, filtering, and amplitude modulation (AM). Readers who are unfamiliar with these topics are suggested to refer to Alexander and Sadiku (2016) or Gilibisco and Monk (2016).

10.4.1 Circuit Analysis

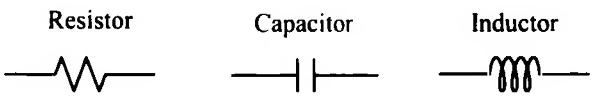


Figure 10.4: Elements of electric circuit

While analyzing circuits, we frequently assume sinusoidal excitation and apply the *phasor technique* (Cheng 1989; Sadiku 2018). For example, *Ohm's law* is written as

$$V(\omega) = I(\omega)\, Z(\omega), \tag{10.7}$$

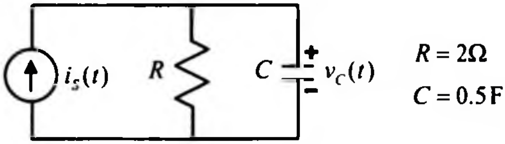
where $V(\omega)$ and $I(\omega)$ represent sinusoidal variations (that vary with $e^{j\omega t}$) of the voltage and current, respectively, and $Z(\omega)$ is the *impedance*. With the phasor technique, the impedances of *resistors*, *capacitors*, and *inductors* are described as

$$Z_R = R, \quad Z_C = \frac{1}{j\omega C}, \quad \text{and} \quad Z_L = j\omega L, \tag{10.8}$$

where R, C, and L represent *resistance*, *capacitance*, and *inductance*, respectively.

Applying Fourier transform to circuit analysis is to generalize the phasor technique. It involves three steps. We first transform the circuit elements into frequency domain and take the Fourier transform of the excitation. Next, we apply circuit techniques such as *Kirchhoff's voltage law* and *Kirchhoff's current law* to find the unknown response (current or voltage). And we finally take the inverse Fourier transform to get the response in the time domain. We should note, however, that Fourier analysis cannot handle circuits with initial conditions. In other words, we only consider circuits that satisfy zero initial conditions.

Example 10.11 Consider the circuit shown below.



The input current to the circuit is given as

$$x(t) = i_s(t) = 10e^{-2t}\,u(t)\ \mathrm{A}.$$

Derive the output voltage at the capacitor: $y(t) = v_C(t)$.

Solution

$$I_s(\omega) = \frac{10}{2 + j\omega},$$

$$Z_R = R = 2, \quad Z_C = \frac{1}{j\omega C} = \frac{2}{j\omega}.$$

Current that flows to the capacitor is then described as

$$I_C(\omega) = \frac{Z_R}{Z_R + Z_C} I_s(\omega) = \frac{2}{2 + 2/(j\omega)}\,\frac{10}{2 + j\omega} = \frac{10j\omega}{(1 + j\omega)(2 + j\omega)}.$$

Output voltage thus becomes

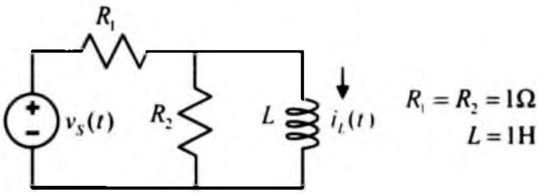
$$V_C(\omega) = I_C(\omega)\,Z_C = \frac{20}{(1 + j\omega)(2 + j\omega)} = \frac{A}{1 + j\omega} + \frac{B}{2 + j\omega},$$

$$\begin{cases} 2A + B = 20, \\ A + B = 0, \end{cases} \quad \rightarrow \quad \begin{cases} A = 20, \\ B = -20, \end{cases}$$

$$V_C(\omega) = \frac{20}{1 + j\omega} - \frac{20}{2 + j\omega},$$

$$y(t) = v_C(t) = 20\,[e^{-t} - e^{-2t}]\,u(t)\ \mathrm{V}.$$

Example 10.12 Consider the circuit shown below.



The input voltage to the circuit is given as

$$x(t) = v_s(t) = 4e^{-t}u(t) \text{ V}.$$

Derive the output current at the inductor: $y(t) = i_L(t)$.

Solution

$$V_s(\omega) = \frac{4}{1 + j\omega},$$

$$Z_{R1} = R_1 = 1, \quad Z_{R2} = R_2 = 1, \quad Z_L = j\omega L = j\omega.$$

Total impedance Z of the circuit is derived as

$$Z = Z_{R1} + \frac{Z_{R2}Z_L}{Z_{R2} + Z_L} = 1 + \frac{j\omega}{1 + j\omega} = \frac{1 + j2\omega}{1 + j\omega}.$$

The current through the first resistor becomes

$$I_{R1}(\omega) = \frac{V_s(\omega)}{Z} = \frac{4}{1 + j2\omega},$$

and thus the output current is

$$\begin{aligned} I_L(\omega) &= \frac{Z_{R2}}{Z_{R2} + Z_L} I_{R1}(\omega) = \frac{1}{1 + j\omega} I_{R1}(\omega) \\ &= \frac{4}{(1 + j\omega)(1 + j2\omega)} = \frac{A}{1 + j\omega} + \frac{B}{1 + j2\omega}, \end{aligned}$$

$$\begin{cases} A + B = 4, \\ 2A + B = 0, \end{cases} \rightarrow \begin{cases} A = -4, \\ B = 8, \end{cases}$$

$$I_L(\omega) = \frac{8}{1+j2\omega} - \frac{4}{1+j\omega} = \frac{4}{1/2+j\omega} - \frac{4}{1+j\omega}.$$

$$y(t) = i_L(t) = 4\,[e^{-t/2} - e^{-t}]\,u(t) \text{ A}.$$

10.4.2 Filtering

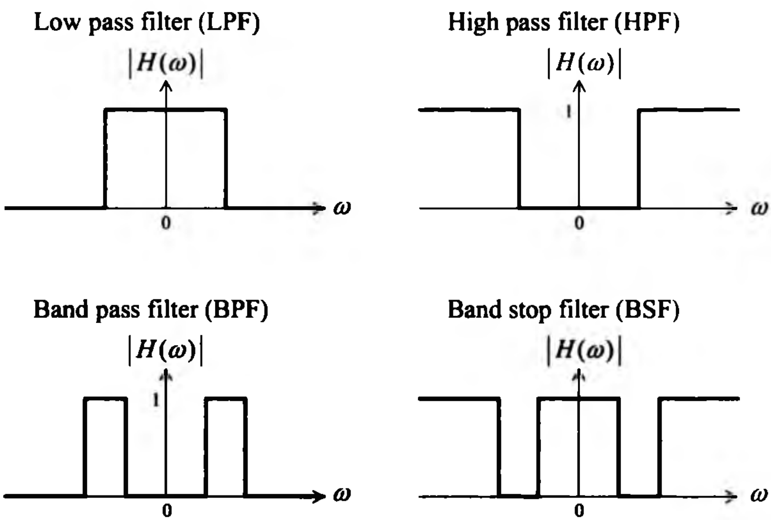


Figure 10.5: Concept of filters

Filters are frequency selective devices, and we design filters to pass signals that contain desired frequency components and block or attenuate others. Fourier analysis of an input signal is thus essential for the filtering process. Figure 10.5 illustrates four types of filters.

- *Low pass filters* (LPF) pass low frequency components and block high frequency components.
- *High pass filters* (HPF) pass high frequency components and block low frequency components.
- *Band pass filters* (BPF) pass frequency components within a frequency band and block frequency components outside the band.

- *Band stop filters* (BSF) pass frequency components outside a frequency band and block frequency components within the band.

Note that while discussing high or low frequency, the sign of frequency does not matter, and we only consider the absolute value of frequency. Note also that all the filters illustrated in Figure 10.5 maintain even symmetries as functions of frequency.

The *RC circuit* shown in Figure 10.6 is a typical of the LPF. Taking the source voltage $v_S(t)$ and capacitor voltage $v_c(t)$ as the input $x(t)$ and output $y(t)$ signals, respectively, we derive the following differential equation:

$$RC\frac{dy(t)}{dt} + y(t) = x(t).$$

The frequency response of the RC circuit is, therefore,

$$H(\omega) = \frac{1}{1 + j\omega RC}. \quad (10.9)$$

Figure 10.7 shows the amplitude and phase spectra of the frequency response $H(\omega)$ one may find with an assumption that the *RC time constant* is 1 second.

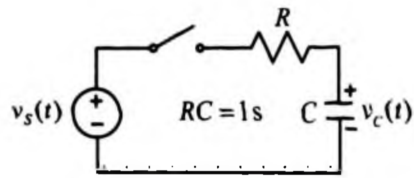


Figure 10.6: An RC circuit whose RC time constant is 1 second

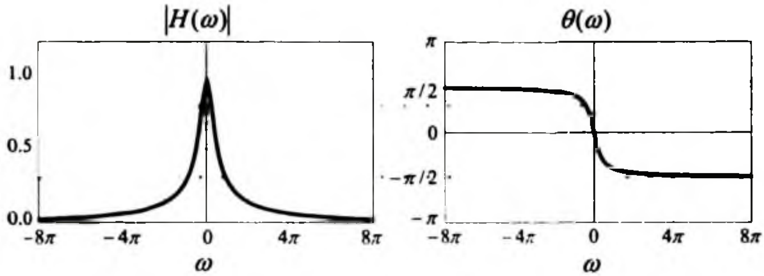


Figure 10.7: Frequency response of the RC circuit shown in Figure 10.6

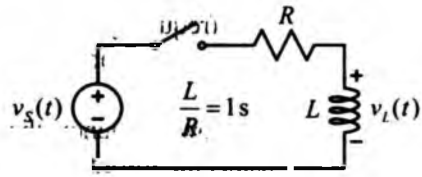


Figure 10.8. An RL circuit whose RL time constant is 1 second

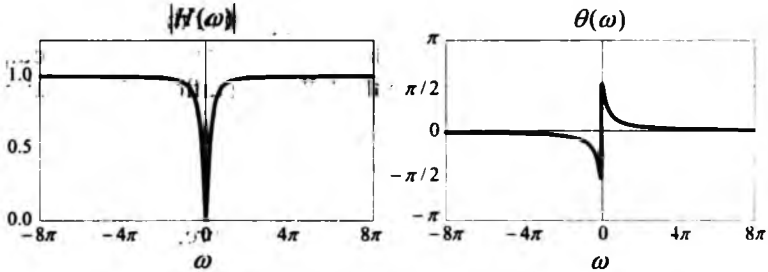


Figure 10.9: Frequency response of the RL circuit shown in Figure 10.8

A typical HPF is the *RL circuit* shown in Figure 10.8. Taking the source voltage $v_s(t)$ and inductor voltage $v_L(t)$ as the input $x(t)$ and output $y(t)$ signals, respectively, we derive the following differential equation:

$$\frac{dy(t)}{dt} + \frac{R}{L}y(t) = \frac{dx(t)}{dt}.$$

The frequency response of the RL circuit is, therefore,

$$H(\omega) = \frac{j\omega L}{R + j\omega L}. \tag{10.10}$$

Figure 10.9 shows the amplitude and phase spectra of the frequency response $H(\omega)$ one may find with an assumption that the *RL time constant* is 1 second.

The above two circuits exemplify continuous-time filters. Filters are, however, not limited to processing continuous-time signals. To the contrary, filters are extensively used for processing discrete-time signals as well. Regardless of the type of signals, however, underlying principles of filters are identical. An example of applying a BPF to a discrete-time signal will be presented in Chapter 12.

10.4.3 Amplitude Modulation

In electronics and telecommunications, *modulation* is the process of varying one or more properties of a periodic waveform (called the *carrier signal*) with a modulating signal that typically contains information to be transmitted. *Demodulation*, on the other hand, is to extract the original information-bearing signal from a carrier wave. Most radio systems in the 20th century used *amplitude modulation* (AM) or *frequency modulation* (FM) for radio broadcast.

Amplitude modulation is a process that we let modulating signals control the amplitude of the carrier. Consider an information signal $x(t)$ and high frequency carrier signal $c(t) = \cos(\omega_0 t)$. Amplitude modulated signal $y(t)$ is expressed in the time domain as

$$y(t) = x(t)\, c(t).$$

Recalling the frequency convolution property of Fourier transform, we express the modulated signal in the frequency domain as

$$\begin{aligned} Y(\omega) = \frac{1}{2\pi} X(\omega) * C(\omega) &= \frac{1}{2} X(\omega) * [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \\ &= \frac{1}{2} [X(\omega + \omega_0) + X(\omega - \omega_0)]. \qquad (10.11) \end{aligned}$$

In other words, the amplitude modulation results in shifting the spectrum of the original signal.

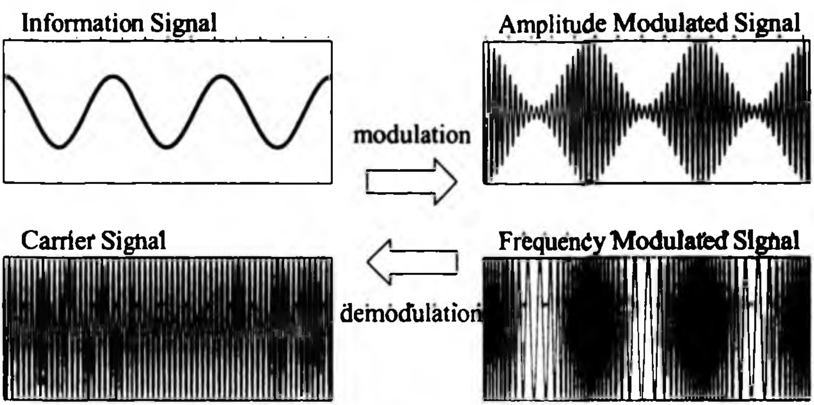


Figure 10.10: Concept of modulation

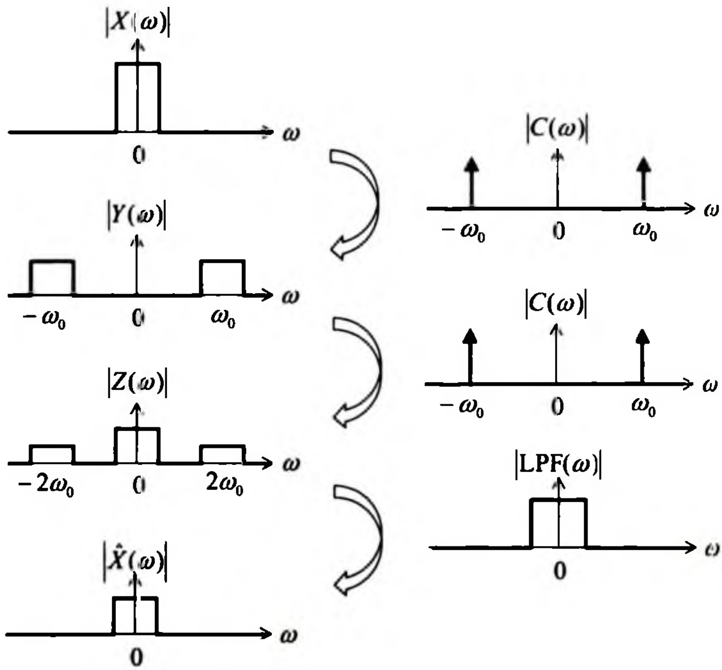


Figure 10.11: Principle of amplitude modulation

At the receiving end of the transmission, the audio information is recovered from the modulated carrier by demodulation that shifts back the message spectrum to its original low frequency location. Consider, for example, multiplying $c(t) = \cos(\omega_0 t)$ once again to the modulated signal $y(t)$ to get the shifted-back signal $z(t)$ such that

$$z(t) = y(t)\,c(t).$$

In frequency domain, the shifted-back signal is expressed as

$$\begin{aligned} Z(\omega) &= \frac{1}{2}\,[Y(\omega+\omega_0) + Y(\omega-\omega_0)] \\ &= \frac{1}{4}\,X(\omega+2\omega_0) + \frac{1}{2}\,X(\omega) + \frac{1}{4}\,X(\omega-2\omega_0). \end{aligned} \tag{10.12}$$

Finally, letting $z(t)$ pass through a low pass filter recovers a copy of the original information signal. The principle of the amplitude modulation is summarized in Figure 10.11.

PROBLEMS

Problem 10.1 Find the inverse Fourier transform of

$$H(\omega) = \frac{4j\omega}{2 + j\omega}.$$

Problem 10.2 Find the inverse Fourier transform of

$$H(\omega) = \frac{2j\omega}{3 + j\omega}.$$

Problem 10.3 Sketch $x(t)$ whose Fourier transform is

$$X(\omega) = \frac{6\sin(4\omega)}{\omega}.$$

Problem 10.4 Sketch $x(t)$ whose Fourier transform is

$$X(\omega) = \frac{4}{1 + j\omega}.$$

Problem 10.5 A continuous-time linear system is described as

$$\frac{dy(t)}{dt} + 3y(t) = x(t).$$

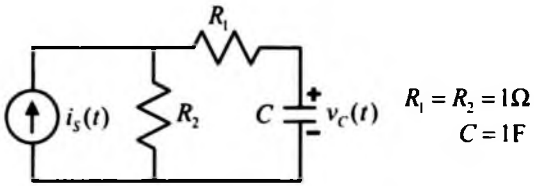
Find the output $y(t)$ due to the input $x(t) = e^{-2t}u(t)$.

Problem 10.6 A continuous-time linear system is described as

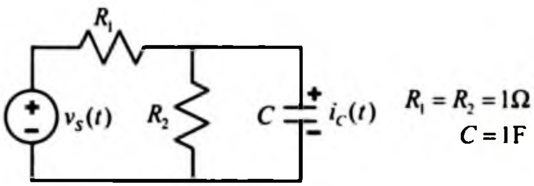
$$\frac{dy(t)}{dt} + y(t) = x(t).$$

Find the output $y(t)$ due to the input $x(t) = e^{-3t}u(t)$.

Problem 10.7 Consider the circuit shown below. The input current is given as $i_s(t) = 8e^{-t}u(t)$ A. Find the output voltage $v_c(t)$ from the circuit.



Problem 10.8 Consider the circuit shown below. The input voltage is given as $v_S(t) = 8e^{-t}u(t)$ V. Find the output current $i_C(t)$ from the circuit.



Problem 10.9 Sketch the amplitude spectrum of a low pass filter.

Problem 10.10 Sketch the amplitude spectrum of a band stop filter.

Chapter 11

FOURIER TRANSFORM OF DT SIGNALS

We have studied that Fourier analysis is a powerful tool of analyzing frequency content of continuous-time signals (Fourier series for periodic functions and Fourier transform for nonperiodic functions). The Fourier analysis also enables one to discuss frequency content of discrete-time signals (discrete-time Fourier transform (DTFT) for nonperiodic sequences and discrete Fourier transform (DFT) for periodic sequences). In Chapter 11, we outline the concept and properties of DTFT and DFT. For more detailed discussion about the Fourier analysis of discrete-time signals, readers can refer to other literature that includes Oppenheim and Willsky (1997), Oppenheim and Schafer (2010), and Lathi and Green (2017).

11.1 DISCRETE-TIME FOURIER TRANSFORM

11.1.1 Concept of Discrete-time Fourier Transform

The Fourier transform of a continuous-time signal $x(t)$ is defined as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

Having a discrete-time signal $x[n]$ instead of a continuous-time signal, one may rely on a logical reasoning and implement the following transform:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}. \tag{11.1}$$

The nature of $X(\omega)$ is not clear yet. However, as was argued in Chapter 2, the frequency function $X(\omega)$ must be periodic in frequency domain such that

$$\begin{aligned} X(\omega + 2\pi) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \cdot e^{-j2\pi n} \\ &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = X(\omega). \end{aligned}$$

In other words, we may regard $X(\omega)$ as a periodic function of frequency that repeats itself at every 2π increase/decrease of angular frequency. We can thus write

$$X(\omega) = X(\omega + 2\pi k), \tag{11.2}$$

where k is an integer.

Now, we further recall that a periodic time function $x(t)$ with fundamental period T_0 is expressed as

$$x(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t},$$

where the fundamental frequency Ω is $2\pi/T_0$, and the Fourier series coefficient $X[m]$ is

$$X[m] = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jm\Omega t}\, dt.$$

Considering that we may safely take opposite sign of Ω, we rewrite the above equations as

$$x(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{-jm\Omega t}, \tag{11.3}$$

and

$$X[m] = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{jm\Omega t}\, dt. \tag{11.4}$$

Note that expression 11.3 is extremely similar to expression 11.1. The similarity illustrates that we may regard expression 11.1 as a Fourier series expansion of the periodic frequency function $X(\omega)$ whose fundamental "angular frequency" period is 2π. We thus adopt the following change of notations:

$T_0 \rightarrow 2\pi,\ \Omega \rightarrow 1,\ x(t) \rightarrow X(\omega),\ X[m] \rightarrow x[n],\ t \rightarrow \omega,$ and $m \rightarrow n,$

and rewrite expressions 11.3 and 11.4 as follows:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\omega}, \tag{11.5}$$

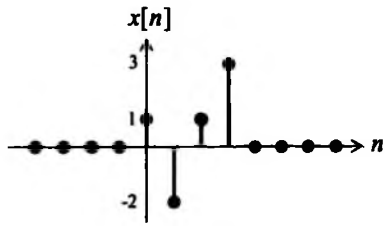
and

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{jn\omega} d\omega. \tag{11.6}$$

Expression 11.5 is the definition of the *discrete-time Fourier transform* (DTFT) of $x[n]$, and expression 11.6 defines the inverse discrete-time Fourier transform of $X(\omega)$. In other words, discrete-time Fourier transform is essentially another expression of Fourier series with time domain and frequency domain interchanging their role.

11.1.2 Examples of Discrete-time Fourier Transform

Example 11.1 Consider the discrete-time Fourier transform of the time sequence shown below.



Derive the real and imaginary parts of $X(\omega)$.

Solution

$$x[n] = \delta[n] - 2\delta[n-1] + \delta[n-2] + 3\delta[n-3].$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\omega} = 1 - 2e^{-j\omega} + e^{-j2\omega} + 3e^{-j3\omega}$$

$$= 1-2(\cos\omega - j\sin\omega) + (\cos 2\omega - j\sin 2\omega) + 3(\cos 3\omega - j\sin 3\omega)$$

$$= [1-2\cos\omega + \cos 2\omega + 3\cos 3\omega] + j[2\sin\omega - \sin 2\omega - 3\sin 3\omega].$$

$$\text{Re}[X(\omega)] = 1-2\cos\omega+\cos 2\omega+3\cos 3\omega,$$

$$\text{Im}[X(\omega)] = 2\sin\omega - \sin 2\omega - 3\sin 3\omega.$$

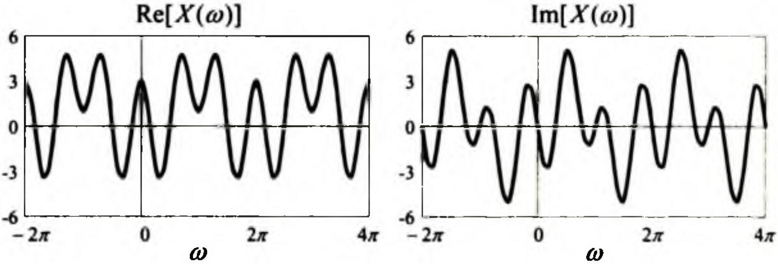


Figure 11.1: Repeating pattern of the discrete-time Fourier transform derived in Example 11.1

Figure 11.1 shows the real and imaginary parts of the discrete-time Fourier transform we derived in Example 11.1. It is obvious that $X(\omega)$ repeats itself every moment ω changes $\pm 2\pi$ (or equivalently, f changes ± 1). That is why we can expand the complex function $X(\omega)$ as a Fourier series and specify its Fourier series coefficients as

$$x[0] = 1, \quad x[1] = -2, \quad x[2] = 1, \quad \text{and} \quad x[3] = 3.$$

Example 11.2 Derive the discrete-time Fourier transform of the following time sequence.

$$x[n] = a^n u[n] \quad (|a| < 1)$$

Solution

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\omega} = \sum_{n=-\infty}^{\infty} a^n u[n]\, e^{-jn\omega} = \sum_{n=0}^{\infty} a^n e^{-jn\omega}$$

$$= \sum_{n=0}^{\infty} (a\, e^{-j\omega})^n = \frac{1}{1 - a\, e^{-j\omega}}.$$

Example 11.3 Derive the discrete-time Fourier transform of the following time sequence.

$$x[n] = a^{|n|} \quad (|a| < 1)$$

Solution

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\omega} = \sum_{n=-\infty}^{\infty} a^{|n|}\, e^{-jn\omega}$$

$$= 1 + \sum_{n=1}^{\infty} a^{n} e^{-jn\omega} + \sum_{n=-1}^{\infty} a^{-n} e^{-jn\omega}$$

$$= 1 + \sum_{n=1}^{\infty} a^{n} e^{-jn\omega} + \sum_{n=1}^{\infty} a^{n} e^{jn\omega}$$

$$= 1 + \sum_{n=1}^{\infty} (a\, e^{-j\omega})^{n} + \sum_{n=1}^{\infty} (a\, e^{j\omega})^{n}$$

$$= 1 + \frac{a\, e^{-j\omega}}{1 - a\, e^{-j\omega}} + \frac{a\, e^{j\omega}}{1 - a\, e^{j\omega}}$$

$$= 1 + \frac{a\, e^{-j\omega} + a\, e^{j\omega} - 2a^2}{1 - a\, e^{-j\omega} - a\, e^{j\omega} + a^2} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}.$$

Example 11.4

$$x[n] = \delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2],$$

$$y[n] = \delta[n+4] + \delta[n+2] + \delta[n] + \delta[n-2] + \delta[n-4],$$

$$z[n] = \delta[n+4] + \delta[n+3] + \delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4].$$

Show that the discrete-time Fourier transforms of the above time sequences are as follows:

$$X(\omega) = \frac{\sin(5\omega/2)}{\sin(\omega/2)}, \quad Y(\omega) = \frac{\sin(5\omega)}{\sin(\omega)}, \quad \text{and} \quad Z(\omega) = \frac{\sin(9\omega/2)}{\sin(\omega/2)}$$

Solution

$$X(\omega) = e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}$$

$$= e^{j2\omega}[1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega}]$$

$$= e^{j2\omega}\frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}} = \frac{e^{j2\omega} - e^{-j3\omega}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\omega/2}\left(e^{j5\omega/2} - e^{-j5\omega/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} = \frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$

$$Y(\omega) = e^{j4\omega} + e^{j2\omega} + 1 + e^{-j2\omega} + e^{-j4\omega}$$

$$= e^{j4\omega}[1 + e^{-j2\omega} + e^{-j4\omega} + e^{-j6\omega} + e^{-j8\omega}]$$

$$= e^{j4\omega}\frac{1 - e^{-j10\omega}}{1 - e^{-j2\omega}} = \frac{e^{j4\omega} - e^{-j6\omega}}{1 - e^{-j2\omega}}$$

$$= \frac{e^{-j\omega}\left(e^{j5\omega} - e^{-j5\omega}\right)}{e^{-j\omega}\left(e^{j\omega} - e^{-j\omega}\right)} = \frac{\sin(5\omega)}{\sin(\omega)}.$$

$$Z(\omega) = e^{j4\omega} + e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega}$$

$$= e^{j4\omega}[1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega} + e^{-j5\omega} + e^{-j6\omega} + e^{-j7\omega} + e^{-j8\omega}]$$

$$= e^{j4\omega}\frac{1 - e^{-j9\omega}}{1 - e^{-j\omega}} = \frac{e^{j4\omega} - e^{-j5\omega}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\omega/2}\left(e^{j9\omega/2} - e^{-j9\omega/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} = \frac{\sin(9\omega/2)}{\sin(\omega/2)}.$$

Figure 11.2 shows the time sequences and their discrete-time Fourier transforms derived in Example 11.4. Comparing $X(\omega)$ with $Y(\omega)$ reveals that $X(\omega)$ and $Y(\omega)$ have a frequency scaling relation such that

$$Y(\omega) = X(2\omega).$$

We recall that for a Fourier transform pair, time expansion of $x(t)$ is associated with frequency compression of $X(\omega)$ as follow:

$$x(t/2) \quad \Leftrightarrow \quad 2X(2\omega).$$

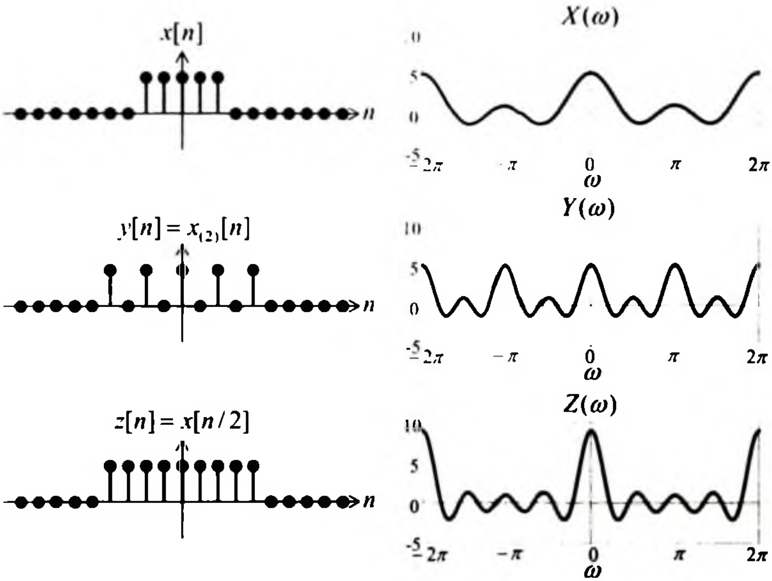


Figure 11.2: Time sequences and their discrete-time Fourier transforms derived in Example 11.4

While treating time sequences, on the other hand, $x[n/2]$ is not associated with $X(2\omega)$. In fact, Figure 11.2 demonstrates that

$$x_{(k)}[n] \quad \Leftrightarrow \quad X(k\omega), \tag{11.7}$$

where $x_{(k)}[n]$ is defined for an integer k as

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k, \\ 0 & \text{if } n \text{ is not a multiple of } k. \end{cases} \tag{11.8}$$

In other words, the number of nonzero elements of a time sequence should be kept constant to satisfy expression 11.7.

11.1.3 Properties of Discrete-time Fourier Transform

Several important properties of the discrete-time Fourier transform are summarized in Table 11.1. Note that many properties we have discussed about Fourier transform (Table 9.1) still applies to discrete-time Fourier transform. There must be, however, distinction between the two. We

Table 11.1: Properties of Discrete-time Fourier transform

Property	$x[n]$	$X(\omega)$
Periodicity	$x[n]$	$X(\omega) = X(\omega + 2k\pi)$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(\omega) + bX_2(\omega)$
Conjugation	$x^*[n]$	$X^*(-\omega)$
Time Reversal	$x[-n]$	$X(-\omega)$
Time Scaling	$x_{(k)}[n]$	$X(k\omega)$
Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(\omega)$
Frequency Shifting	$e^{j\omega_0 n} x[n]$	$X(\omega - \omega_0)$
Time Differencing	$x[n] - x[n-1]$	$(1 - e^{-j\omega})\, X(\omega)$
Time convolution	$x_1[n] * x_2[n]$	$X_1(\omega)\, X_2(\omega)$
Frequency convolution	$x_1[n]\, x_2[n]$	$\frac{1}{2\pi} X_1(\omega) \circledast X_2(\omega)$

have discussed, for example, that time scaling can be associated with frequency scaling only if the time scaling satisfies the condition described in expression 11.8.

Consider also frequency convolution. Multiplying two time sequences is associated with taking convolution in frequency domain. The convolution in frequency domain is, however, not a kind of convolution we have discussed so far. Denoting $y[n] = x_1[n]x_2[n]$, we describe the discrete-time Fourier transform of $y[n]$ as

$$\begin{aligned} Y(\omega) &= \sum_{n=-\infty}^{\infty} y[n]\, e^{-jn\omega} = \sum_{n=-\infty}^{\infty} x_1[n]\, x_2[n]\, e^{-jn\omega} \\ &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\varphi)\, e^{jn\varphi} d\varphi \right] x_2[n]\, e^{-jn\omega} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\varphi) \left[\sum_{n=-\infty}^{\infty} e^{jn\varphi} x_2[n]\, e^{-jn\omega} \right] d\varphi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\varphi) \left[\sum_{n=-\infty}^{\infty} x_2[n]\, e^{-jn(\omega-\varphi)} \right] d\varphi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\varphi)\, X_2(\omega - \varphi)\, d\varphi. \end{aligned}$$

The last integral of the above expression defines the *circular convolution integral*, which we denote as

$$X_1(\omega) \circledast X_2(\omega) = \int_{0}^{2\pi} X_1(\varphi)\, X_2(\omega - \varphi)\, d\varphi. \qquad (11.9)$$

And using the circular convolution notation, we express the frequency convolution property as

$$x_1[n]\, x_2[n] \quad \Leftrightarrow \quad \frac{1}{2\pi} X_1(\omega) \circledast X_2(\omega). \qquad (11.10)$$

Example 11.5 Find the inverse discrete-time Fourier transform of the following frequency function.

$$Y(\omega) = \frac{1}{(1 - ae^{-j\omega})^2} \quad (|a| < 1).$$

Solution

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}},$$

$$x[n] = a^n u[n] \quad (|a| < 1),$$

$$Y(\omega) = X(\omega)\, X(\omega),$$

$$\begin{aligned} y[n] &= x[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]\, x[n-k] \\ &= \sum_{k=-\infty}^{\infty} a^k u[k]\, a^{n-k} u[n-k] = \left(\sum_{k=0}^{n} a^k a^{n-k}\right) u[n] \\ &= \left(\sum_{k=0}^{n} a^n\right) u[n] = a^n (n+1)\, u[n]. \end{aligned}$$

11.2 DISCRETE FOURIER TRANSFORM

11.2.1 Concept of Discrete Fourier Transform

Discrete-time Fourier transform defines Fourier transform of nonperiodic sequences. It is *discrete Fourier transform* (DFT) that defines Fourier transform of periodic sequences. We first denote a periodic time sequence as

$$x[n] = \{x[0],\ x[1],\ x[2],\ \cdots,\ x[N-1]\},$$

where N denotes the period of the sequence. Note that the period N can be a multiple of the fundamental period N_0. Discrete Fourier transform associates the periodic time sequence $x[n]$ with a frequency domain sequence $X[m]$ that is also periodic in frequency domain with the identical period N such that

$$X[m] = \{X[0],\ X[1],\ X[2],\ \cdots,\ X[N-1]\}.$$

More specifically, the two sequences are related as

$$X[m] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi mn/N}. \tag{11.11}$$

and

$$x[n] = \frac{1}{N}\sum_{m=0}^{N-1} X[m]\, e^{j2\pi mn/N}. \tag{11.12}$$

Expression 11.11 is the definition of the discrete Fourier transform of $x[n]$, and expression 11.12 defines the inverse discrete Fourier transform of $X[m]$. The periodicity of $x[n]$ and $X[m]$ can be demonstrated as follows:

$$\begin{aligned} x[n+kN] &= \frac{1}{N}\sum_{m=0}^{N-1} X[m]\, e^{j2\pi m(n+kN)/N} = \frac{1}{N}\sum_{m=0}^{N-1} X[m]\, e^{j2\pi mn/N}\, e^{j2\pi mk} \\ &= \frac{1}{N}\sum_{m=0}^{N-1} X[m]\, e^{j2\pi mn/N} = x[n], \end{aligned}$$

and

$$\begin{aligned} X[m+kN] &= \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi(m+kN)n/N} = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi mn/N}\, e^{-j2\pi kn} \\ &= \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi mn/N} = X[m], \end{aligned}$$

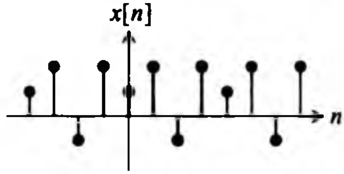
where k is an integer.

11.2.2 Examples of Discrete Fourier Transform

Example 11.6 Sketch the following time sequence and derive the discrete Fourier transform of it.

$$x[n] = \{1,\ 2,\ -1,\ 2\}.$$

Solution



$$X[m] = \sum_{n=0}^{3} x[n]\, e^{-j2\pi mn/4}$$
$$= x[0] + x[1]\, e^{-j\pi m/2} + x[2]\, e^{-j\pi m} + x[3]\, e^{-j3\pi m/2}$$
$$= 1 + 2e^{-j\pi m/2} - e^{-j\pi m} + 2e^{-j3\pi m/2}.$$

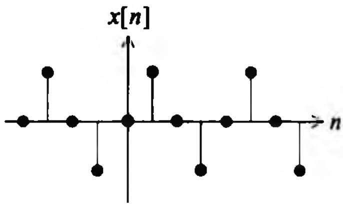
$$X[0] = 1 + 2 - 1 + 2 = 4,$$
$$X[1] = 1 + 2e^{-j\pi/2} - e^{-j\pi} + 2e^{-j3\pi/2} = 1 - 2j + 1 + 2j = 2,$$
$$X[2] = 1 + 2e^{-j\pi} - e^{-j2\pi} + 2e^{-j3\pi} = 1 - 2 - 1 - 2 = -4,$$
$$X[3] = 1 + 2e^{-j3\pi/2} - e^{-j3\pi} + 2e^{-j9\pi/2} = 1 + 2j + 1 - 2j = 2.$$

$$X[m] = \{4,\ 2,\ -4,\ 2\}.$$

Example 11.7 Sketch the following time sequence and derive the discrete Fourier transform of it.

$$x[n] = \{0,\ 2,\ 0,\ -2\}.$$

Solution



$$X[m] = \sum_{n=0}^{3} x[n]\, e^{-j2\pi mn/4}$$
$$= x[0] + x[1]\, e^{-j\pi m/2} + x[2]\, e^{-j\pi m} + x[3]\, e^{-j3\pi m/2}$$
$$= 2e^{-j\pi m/2} - 2e^{-j3\pi m/2}.$$

$$X[0] = 2 - 2 = 0,$$
$$X[1] = 2e^{-j\pi/2} - 2e^{-j3\pi/2} = -2j - 2j = -4j,$$
$$X[2] = 2e^{-j\pi} - 2e^{-j3\pi} = -2 + 2 = 0,$$
$$X[3] = 2e^{-j3\pi/2} - 2e^{-j9\pi/2} = 2j + 2j = 4j.$$

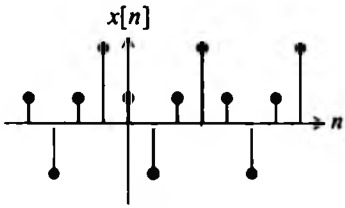
$$X[m] = \{0, \ -4j, \ 0, \ 4j\}.$$

Examples 11.6 and 11.7 demonstrate that the even symmetry of $x[n]$ accompanies real-valued $X[m]$, whereas the odd symmetry of $x[n]$ brings about $X[m]$ that is zero or imaginary.

Example 11.8 Sketch the following time sequence and derive the discrete Fourier transform of it.

$$x[n] = \{1, \ -2, \ 1, \ 3\}.$$

Solution



$$X[m] = \sum_{n=0}^{3} x[n]\, e^{-j2\pi mn/4}$$

$$= x[0] + x[1]\, e^{-j\pi m/2} + x[2]\, e^{-j\pi m} + x[3]\, e^{-j3\pi m/2}$$

$$= 1 - 2e^{-j\pi m/2} + e^{-j\pi m} + 3e^{-j3\pi m/2}.$$

$$X[0] = 1 - 2 + 1 + 3 = 3,$$

$$X[1] = 1 - 2e^{-j\pi/2} + e^{-j\pi} + 3e^{-j3\pi/2} = 1 + 2j - 1 + 3j = 5j,$$

$$X[2] = 1 - 2e^{-j\pi} + e^{-j2\pi} + 3e^{-j3\pi} = 1 + 2 + 1 - 3 = 1,$$

$$X[3] = 1 - 2e^{-j3\pi/2} + e^{-j3\pi} + 3e^{-j9\pi/2} = 1 - 2j - 1 - 3j = -5j.$$

$$X[m] = \{3,\ 5j,\ 1,\ -5j\}.$$

Figure 11.3 shows the real and imaginary parts of the discrete Fourier transform derived in Example 11.8. Note that Figure 11.3 also shows the graphs of Figure 11.1 and highlights the connection between the discrete-time Fourier transform (DTFT) and discrete Fourier transform (DFT).

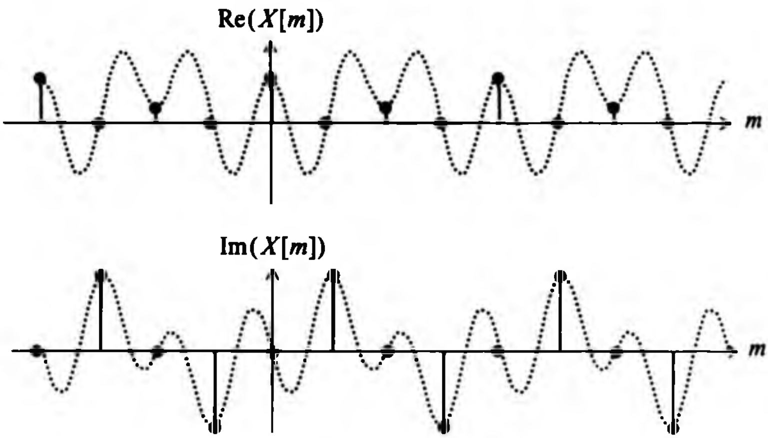


Figure 11.3: Discrete Fourier transform (DFT) derived in Example 11.8. The period of $X[m]$ is 4. Dotted curves represent discrete-time Fourier transform (DTFT) derived in Example 11.1.

Dotted curves indicate the DTFT of the following nonperiodic sequence:

$$\delta[n] - 2\delta[n-1] + \delta[n-2] + 3\delta[n-3],$$

while discrete plots indicate the DFT of the following periodic sequence:

$$\{1, \ -2, \ 1, \ 3\}.$$

Figure 11.3 illustrates that discrete Fourier transform (DFT) is, essentially, equivalent to "sampling" discrete-time Fourier transform (DTFT) at the following uniformly spaced frequencies :

$$\omega = 2\pi m/N,$$

where $m = 0, \ 1, \ 2, \ \cdots, \ N-1$.

Example 11.9 Derive the inverse discrete Fourier transform of the following periodic sequence:

$$X[m] = \{3, \ 5j, \ 1, \ -5j\}.$$

Solution

$$\begin{aligned} x[n] &= \frac{1}{4}\sum_{m=0}^{3} X[m]\, e^{j2\pi mn/4} \\ &= \frac{1}{4}X[0] + \frac{1}{4}X[1]\, e^{j\pi n/2} + \frac{1}{4}X[2]\, e^{j\pi n} + \frac{1}{4}X[3]\, e^{j3\pi n/2} \\ &= \frac{3}{4} + \frac{5j}{4}\, e^{j\pi n/2} + \frac{1}{4}\, e^{j\pi n} - \frac{5j}{4}\, e^{j3\pi n/2}. \end{aligned}$$

$$X[0] = \frac{3}{4} + \frac{5j}{4} + \frac{1}{4} - \frac{5j}{4} = 1,$$

$$X[1] = \frac{3}{4} + \frac{5j}{4}\, e^{j\pi/2} + \frac{1}{4}\, e^{j\pi} - \frac{5j}{4}\, e^{j3\pi/2} = \frac{3}{4} - \frac{5}{4} - \frac{1}{4} - \frac{5}{4} = -2,$$

$$X[2] = \frac{3}{4} + \frac{5j}{4}\, e^{j\pi} + \frac{1}{4}\, e^{j2\pi} - \frac{5j}{4}\, e^{j3\pi} = \frac{3}{4} - \frac{5j}{4} + \frac{1}{4} + \frac{5j}{4} = 1,$$

$$X[3] = \frac{3}{4} + \frac{5j}{4}\, e^{j3\pi/2} + \frac{1}{4}\, e^{j3\pi} - \frac{5j}{4}\, e^{j9\pi/2} = \frac{3}{4} + \frac{5}{4} - \frac{1}{4} + \frac{5}{4} = 3.$$

$$x[n] = \{1, \ -2, \ 1, \ 3\}.$$

For $N = 4$, discrete Fourier transform is summarized in a matrix form as follows:

$$X[m] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi mn/N} = \sum_{n=0}^{3} x[n]\, e^{-j\pi mn/2},$$

$$X[0] = \sum_{n=0}^{3} x[n] = x[0] + x[1] + x[2] + x[3],$$

$$X[1] = \sum_{n=0}^{3} x[n]\, e^{-j\pi n/2} = x[0] + x[1]\, e^{-j\pi/2} + x[2]\, e^{-j\pi} + x[3]\, e^{-j3\pi/2},$$

$$X[2] = \sum_{n=0}^{3} x[n]\, e^{-j\pi n} = x[0] + x[1]\, e^{-j\pi} + x[2]\, e^{-j2\pi} + x[3]\, e^{-j3\pi},$$

$$X[3] = \sum_{n=0}^{3} x[n]\, e^{-j3\pi n/2} = x[0] + x[1]\, e^{-j3\pi/2} + x[2]\, e^{-j3\pi} + x[3]\, e^{-j9\pi/2},$$

$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{pmatrix}.$$

Likewise, for $N = 4$, inverse discrete Fourier transform is summarized as

$$x[n] = \frac{1}{N} \sum_{m=0}^{N-1} X[m]\, e^{j2\pi mn/N} = \frac{1}{4} \sum_{m=0}^{3} X[m]\, e^{j\pi mn/2},$$

$$x[0] = \frac{1}{4} \sum_{m=0}^{3} X[m] = \frac{X[0]}{4} + \frac{X[1]}{4} + \frac{X[2]}{4} + \frac{X[3]}{4},$$

$$x[1] = \frac{1}{4} \sum_{m=0}^{3} X[m]\, e^{j\pi m/2} = \frac{X[0]}{4} + \frac{X[1]}{4} e^{j\pi/2} + \frac{X[2]}{4} e^{j\pi} + \frac{X[3]}{4} e^{j3\pi/2},$$

$$x[2] = \frac{1}{4} \sum_{m=0}^{3} X[m]\, e^{j\pi m} = \frac{X[0]}{4} + \frac{X[1]}{4} e^{j\pi} + \frac{X[2]}{4} e^{j2\pi} + \frac{X[3]}{4} e^{j3\pi},$$

$$x[3] = \frac{1}{4} \sum_{m=0}^{3} X[m]\, e^{j3\pi m/2} = \frac{X[0]}{4} + \frac{X[1]}{4} e^{j3\pi/2} + \frac{X[2]}{4} e^{j3\pi} + \frac{X[3]}{4} e^{j9\pi/2}.$$

$$\begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{pmatrix}.$$

And the multiplication of the two matrices that represent DFT and inverse DFT, respectively, yields the 4×4 identity matrix as follow:

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In general, DFT and inverse DFT are summarized in the following matrix equations:

$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & z^{-1} & z^{-2} & \cdots & z^{-(N-1)} \\ 1 & z^{-2} & z^{-4} & \cdots & z^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z^{-(N-1)} & z^{-2(N-1)} & \cdots & z^{-(N-1)^2} \end{pmatrix}\begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{pmatrix}, \quad (11.13)$$

and

$$\begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{pmatrix} = \frac{1}{N}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & z & z^{2} & \cdots & z^{(N-1)} \\ 1 & z^{2} & z^{4} & \cdots & z^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z^{(N-1)} & z^{2(N-1)} & \cdots & z^{(N-1)^2} \end{pmatrix}\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{pmatrix}, \quad (11.14)$$

where $z = e^{j2\pi/N}$.

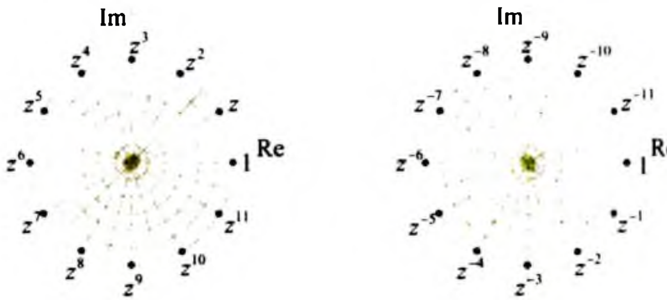
Example 11.10 Consider the discrete Fourier transform of the following periodic sequence:

$$x[n] = \{1,\ 1,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 1\}.$$

What is $X[11]$?

Solution

We count that $N = 12$ and denote $z = e^{j2\pi/N} = e^{j\pi/6}$. We also sketch powers of z and derive $X[11]$ as follows:



$$\begin{aligned} X[11] &= x[0] + z^{-11}x[1] + z^{-33}x[3] + z^{-66}x[6] + z^{-121}x[11] \\ &= 1 + z^{-11} + z^{-33} + z^{-66} + z^{-121} \\ &= 1 + z + z^3 + z^6 + z^{11} = \sqrt{3} + j \end{aligned}$$

11.2.3 Properties of Discrete Fourier Transform

Several important properties of discrete Fourier transform are summarized in Table 11.2. Note that many properties we have discussed about Fourier transform (Table 9.1) and discrete-time Fourier transform (Table 11.1) still apply to discrete-time Fourier transform. There are, however, properties that are unique to discrete Fourier transform. Consider, for example. the time convolution property. Denoting $Y[m] = X_1[m]\,X_2[m]$, we describe the inverse discrete Fourier transform of $Y[m]$ as

$$y[n] = \frac{1}{N}\sum_{m=0}^{N-1} Y[m]\,e^{j2\pi mn/N} = \frac{1}{N}\sum_{m=0}^{N-1} X_1[m]\,X_2[m]\,e^{j2\pi mn/N}$$

Table 11.2: Properties of Discrete Fourier transform

Property	$x[n]$	$X[m]$
Periodicity	$x[n] = x[n + kN]$	$X[m] = X[m + kN]$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1[m] + bX_2[m]$
Conjugation	$x^*[n]$	$X^*[-m]$
Time Reversal	$x[-n]$	$X[-m]$
Time Shifting	$x[n - n_0]$	$e^{-j2\pi m n_0/N} X[m]$
Frequency Shifting	$e^{j2\pi m_0 n/N} x[n]$	$X[m - m_0]$
Time convolution	$x_1[n] \circledast x_2[n]$	$X_1[m]\, X_2[m]$
Frequency convolution	$x_1[n]\, x_2[n]$	$\frac{1}{N} X_1[m] \circledast X_2[m]$

$$= \frac{1}{N} \sum_{m=0}^{N-1} \left(\sum_{k=0}^{N-1} x_1[k]\, e^{-j2\pi mk/N} \right) X_2[m]\, e^{j2\pi mn/N}$$

$$= \sum_{k=0}^{N-1} x_1[k] \left(\frac{1}{N} \sum_{m=0}^{N-1} e^{-j2\pi mk/N}\, X_2[m]\, e^{j2\pi mn/N} \right)$$

$$= \sum_{k=0}^{N-1} x_1[k] \left(\frac{1}{N} \sum_{m=0}^{N-1} X_2[m]\, e^{j2\pi m(n-k)/N} \right) = \sum_{k=0}^{N-1} x_1[k]\, x_2[n-k]$$

The last sum of the above expression defines the *circular convolution sum*, which we denote as

$$x_1[n] \circledast x_2[n] = \sum_{k=0}^{N-1} x_1[k]\, x_2[n-k]. \tag{11.15}$$

And using the circular convolution notation, we express the time convolution property as

$$x_1[n] \circledast x_2[n] \quad \Leftrightarrow \quad X_1[m]\, X_2[m]. \tag{11.16}$$

11.3 SUMMARY OF FOURIER ANALYSIS

Fourier series

$$x(t) = \sum_{m=-\infty}^{\infty} X[m]\, e^{jm\Omega t} \qquad X[m] = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jm\Omega t}\, dt$$

$$x(t) = x(t + kT_0)$$

Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{jn\omega}\, d\omega \qquad X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\omega}$$

$$X(\omega) = X(\omega + 2\pi k)$$

Discrete Fourier transform

$$x[n] = \frac{1}{N}\sum_{m=0}^{N-1} X[m]\, e^{j2\pi mn/N} \qquad X[m] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi mn/N}$$

$$x[n] = x[n + kN] \qquad X[m] = X[m + kN]$$

PROBLEMS

Problem 11.1 Determine time sequence $x[n]$ that yields the following discrete-time Fourier transform:

$$X(\omega) = 1 + 3e^{-j2\omega} - 2e^{-j3\omega} + e^{-j4\omega}.$$

Problem 11.2 Determine time sequence $x[n]$ that yields the following discrete-time Fourier transform:

$$X(\omega) = 2 - e^{-j2\omega} + 3e^{-j3\omega} - 2e^{-j4\omega}.$$

Problem 11.3 Consider the following time sequence:

$$x[n] = \delta[n] + 2\delta[n-1] - 3\delta[n-2].$$

Derive the real and imaginary parts of the discrete-time Fourier transform $X(\omega)$, respectively.

Problem 11.4 Consider the following time sequence:

$$x[n] = \delta[n] - 3\delta[n-1] + 2\delta[n-2].$$

Derive the real and imaginary parts of the discrete-time Fourier transform $X(\omega)$, respectively.

Problem 11.5 Find the discrete Fourier transform $X[m]$ of the following periodic sequence:

$$x[n] = \{2,\ -1,\ 1,\ -1\}.$$

Problem 11.6 Find the discrete Fourier transform $X[m]$ of the following periodic sequence:

$$x[n] = \{0,\ -2,\ 0,\ 2\}.$$

Problem 11.7 Find the inverse discrete Fourier transform $x[n]$ of the following periodic sequence:

$$X[m] = \{1, \ 1, \ 5, \ 1\}.$$

Problem 11.8 Find the inverse discrete Fourier transform $x[n]$ of the following periodic sequence:

$$X[m] = \{0, \ 4j, \ 0, \ -4j\}.$$

Problem 11.9 Consider the discrete Fourier transform $X[m]$ of the following periodic sequence:

$$x[n] = \{1, \ 1, \ 0, \ 1, \ 0, \ 0, \ 1, \ 0, \ 0, \ 0, \ 0, \ 1\}.$$

What is $X[2]$?

Problem 11.10 Consider the discrete Fourier transform $X[m]$ of the following periodic sequence:

$$x[n] = \{1, \ 1, \ 0, \ 1, \ 0, \ 0, \ 1, \ 0, \ 0, \ 0, \ 0, \ 1\}.$$

What is $X[3]$?

Chapter 12

NUMERICAL EXERCISE VIA FFT

Chapters 7 through 11 have provided discussion about Fourier analysis of different signal types. As the final step of Fourier analysis, we focus on data sampling and numerical implementation of Fourier transform via digital computers. The numerical work is based on the MATLAB package. Those who are not familiar with MATLAB are recommended to review Chapter 6 or other literature that includes Hahn and Valentine (2019) and Moore (2014).

12.1 SAMPLING PERIOD AND DURATION

Sampling is an important operation in signal processing. It may be regarded as a way of converting continuous-time signals to discrete-time signals. In other words, sampling is the bridge from continuous-time to discrete-time signals. Sampling can be done by using a train of impulses. We should note, however, that sampling can introduce a loss of information.

Consider sampling (or measuring) data 20 times a second. The *sampling frequency* f_s is thus 20 Hz, and the *sampling period* (or interval) Δt is 1/20 second. With the f_s value, one may safely argue that there is no problem detecting 2 Hz signals. On the other hand, a data set whose sampling frequency is 20 Hz does not provide any significant information about phenomena that vary harmonically one million times a second. What then is the highest frequency that a data set with f_s = 20 Hz may deliver meaningful information?

Figure 12.1 provides a qualitative answer to the question. Four different cases are shown with the identical sampling frequency (f_s = 20). Gray curves represent the original data we try to measure, and the original data frequencies are 2 Hz, 4 Hz, 8 Hz, and 16 Hz for Figure 12.1 (a) through (d), respectively. Black lines depict what it looks like when we simply connect what we have measured. It is evident that up to 8 Hz, we are able to identify the original data frequency. In case of Figure 12.1 (d), however, it is obvious we are unable to detect 16 Hz data and the measured

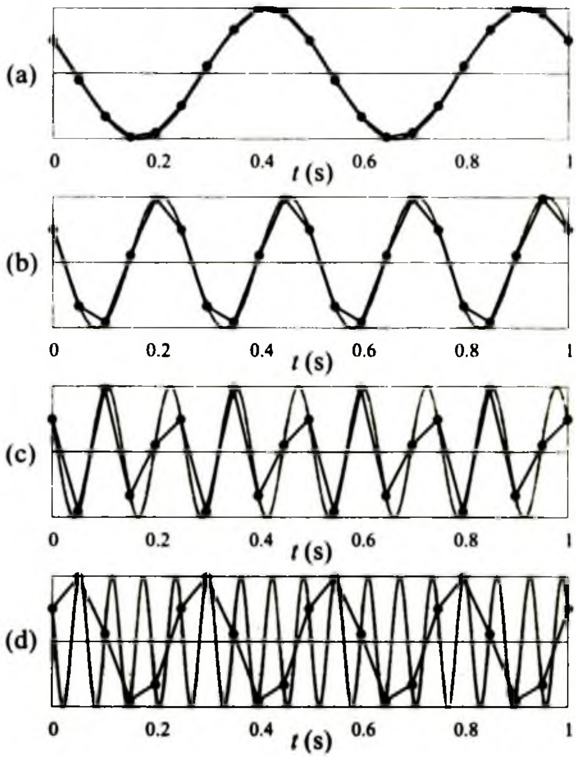


Figure 12.1: Signal frequency (gray curves) versus sampling frequency (black dots). Signal frequency values are (a) 2 Hz, (b) 4 Hz, (c) 8 Hz, and (d) 16 Hz, respectively.

data apparently varies with 4 Hz frequency. In fact, the 20 Hz sampling frequency ensures one to identify only upto 10 Hz signal, and, with a given sampling frequency f_s, the highest measurable frequency f_{max} is the half of the sampling frequency such that

$$f_{max} = \frac{f_s}{2} = \frac{1}{2\Delta t}. \tag{12.1}$$

Expression 12.1 is important while processing an already-acqured data set. There are, one the other hand, situations that one need to decide the sampling frequency before the outset of an experiment. In such a situation, one has to ask what the target frequency range of the experiment is. Denote the target frequency range as

$$f_L \leq f \leq f_U.$$

where f_L and f_U represent the lower and upper limit of the target frequency range. The minimum sampling frequency then should be

$$f_s \geq 2f_U. \tag{12.2}$$

and we call the minimum value of the sampling frequency the *Nyquist frequency*.

A question that naturally follows the above argument is how we do ensure the lower frequency limit f_L of the target frequency range. A quick answer is that we have to measure for sufficiently long time. In other words, the lower frequency limit is related to the *sampling duration* that we express as

$$0 \leq t \leq t_{\max}.$$

Figure 12.2 shows four different cases that measure signals for 1 second ($t_{\max} = 1$). It is evident that it is unlikely to identify signals whose periods are longer than the maximum time window $t_{\max}$. We can thus argue that with a given $t_{\max}$ value, the lowest measurable frequency $f_{\min}$ is given as

$$f_{\min} = \frac{1}{t_{\max}}. \tag{12.3}$$

And we also argue that while deciding the sampling duration before an experiment, one has to make sure to measure for long enough time such that

$$t_{\max} \geq \frac{1}{f_L}. \tag{12.4}$$

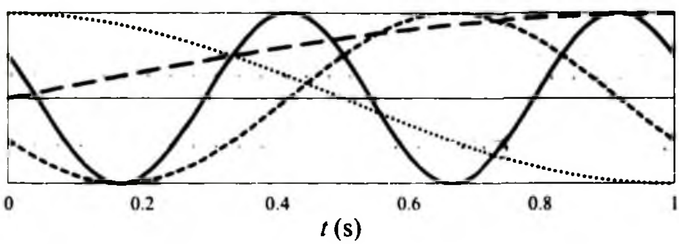


Figure 12.2: Signal frequency versus sampling duration

12.2 WHAT IS FAST FOURIER TRANSFORM?

Most of the time, using a digital computer for Fourier transform means doing discrete Fourier transform (DFT). And doing a discrete Fourier transform is to solve the matrix equations in expressions 11.13 and 11.14. For periodic sequences whose period N is small, it is trivial to solve the matrix equations. As the period N increases, however, the computation of DFT quickly becomes a demanding task, because the number of multiplications necessary for a DFT increases with N^2.

There have been efforts to ease the calculation of DFT. Among them, the most well known and widely used algorithm is called the *fast Fourier transform* (FFT), which was first introduced by Cooley and Tukey (1965). In other words, the FFT is an algorithm that performs DFT with a great computational efficiency. In this study, we do not join discussing computer algorithms but provide guidelines of using the FFT. For the most part, readers are safe to regard the DFT and FFT identically. More in depth discussion about the FFT is provided by Brigham (1988).

We have frequently emphasized that DFT (and FFT as well) is for periodic sequences. Most of the data sets we intend to do Fourier transform are, however, not periodic sequences. We should therefore remember that doing a DFT with a nonperiodic sequence is essentially accepting an assumption that the nonperiodic data set is repeating itself as a whole. In other words, a DFT algorithm always regards a finite sequence that we provide as a periodic sequence and also regards the length N of the finite sequence as the period N of the periodic sequence.

12.2.1 FFT Exercise with Handel

For the first exercise of the FFT, we use a data set that comes with MATLAB. Type the following commands within the MATLAB Command Window:

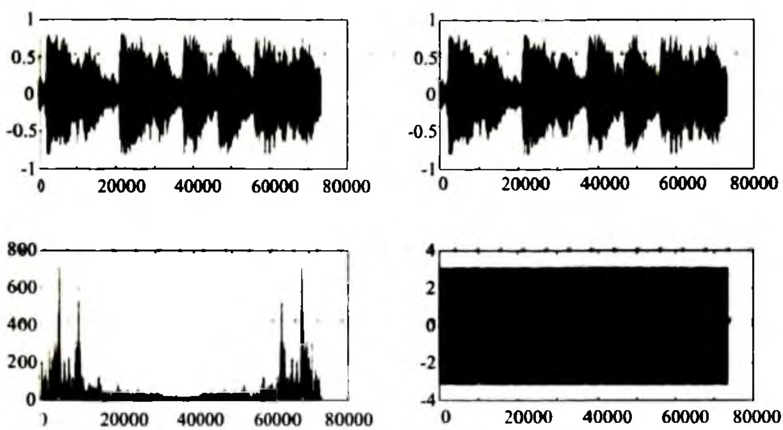
```
load handel;
soundsc(y,Fs);
```

One should hear the beautiful chorus "Hallelujah", which decorates the climax of Handel's oratorio "Messiah". One should also notice that the MATLAB Workspace shows two variables. The first variable Fs indicates the sampling frequency, and its value is 8192 (i.e., 8 kHz). Another variable y represents an array that contains 73113 real numbers, and those numbers altogether constitute the audio signal of the chorus "Hallelujah".

Example 12.1 Type the following MATLAB script and run it.

```
clear;
load handel;
xt = y;
xf = fft(xt);              % Fast Fourier transform
xt2 = ifft(xf);            % Inverse Fast Fourier transform
xf_ab = abs(xf);
xf_ph = angle(xf);

figure(1);
subplot(2,2,1);
plot(xt);
subplot(2,2,2);
plot(xt2);
subplot(2,2,3);
plot(xf_ab);
subplot(2,2,4);
plot(xf_ph,'.');
```

Solution



Example 12.1 demonstrates that MATLAB functions `fft` and `ifft` do perform the DFT and inverse DFT. The first and second subplots exhibit that doing the DFT and inverse DFT in succession does not alter the signal and preserves the original signal as it is. And the third and fourth subplots are the amplitude and phase spectra of the audio signal: the phase spectrum looks enigmatic but the amplitude spectrum, on the other hand, clearly shows amplitude peaks. Interestingly, the amplitude spectrum also shows a symmetric pattern, but, in general, it is difficult to take any quantitative information from the subplots of Example 12.1. The main reason for the difficulty is because the horizontal axes only show the sequence indexes n or m and do not show time or frequency scales. And it is up to a user to make those scales. DFT or FFT do not do that.

The sampling frequency f_s and total sample number N of "Hallelujah" are

$$f_s = 8192 \quad \text{and} \quad N = 73113,$$

respectively. The sampling period Δt and sampling duration $t_{\max}$ are thus

$$\Delta t = 1/f_s = 1/8192 \quad \text{and} \quad t_{\max} = (N-1)\Delta t = 73112/8192,$$

respectively. At the beginning of Chapter 12, we have discussed that with a given sampling period and sampling duration, we may determine the highest and lowest frequencies that a data set can deliver meaningful information. In case of "Hallelujah", those frequency values are determined as

$$f_{\max} = \frac{1}{2\Delta t} = 4096 \quad \text{and} \quad f_{\min} = \frac{1}{t_{\max}} = \frac{8192}{73112},$$

respectively. Note that the $f_{\min}$ value is, in fact, equivalent to the "frequency" sampling period (or interval) Δf. Recall also that DFT associates two sequences $x[n]$ and $X[m]$ that have an identical sample number N. N frequency samples thus represent frequency values within the following range:

$$0 \le f \le (N-1)\Delta f = (N-1) f_{\min} = \frac{N-1}{t_{\max}} = \frac{1}{\Delta t} = 2 f_{\max}.$$

One may be puzzled to notice that the frequency range reaches beyond the $f_{\max}$ value (the highest frequency that our data set can deliver meaningful

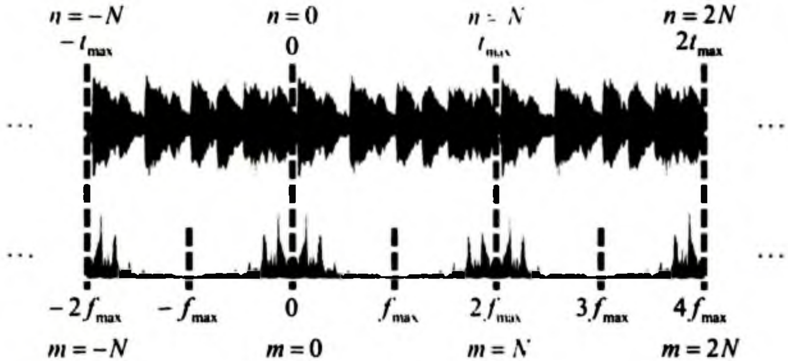


Figure 12.3: Periodicity related to the discrete Fourier transform of "Hallelujah"

information). It is, in fact, not a puzzling thing at all, and we only need to recall that DFT regards both the time sequence $x[n]$ and frequency sequence $X[m]$ as periodic sequences with the period N. All the above argument can be thus summarized in Figure 12.3.

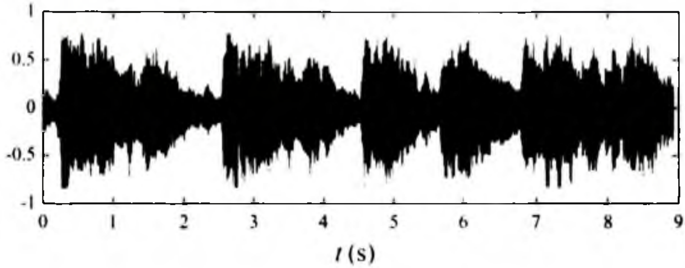
Figure 12.3 shows the genuine periodic sequences DFT is handling. It is evident that the frequency domain data between f_{max} and $2f_{max}$ corresponds to the data between $-f_{max}$ and zero frequency. Utilizing the above time and frequency axes information, we are ready to do Example 12.2.

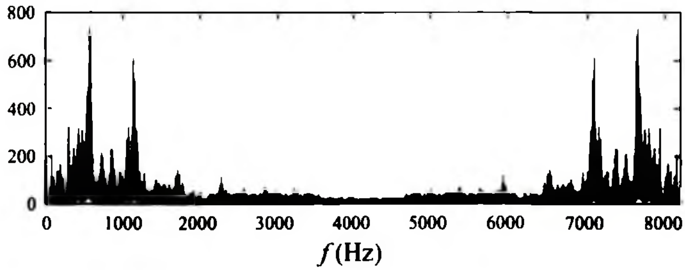
Example 12.2 Type the following MATLAB script and run it.

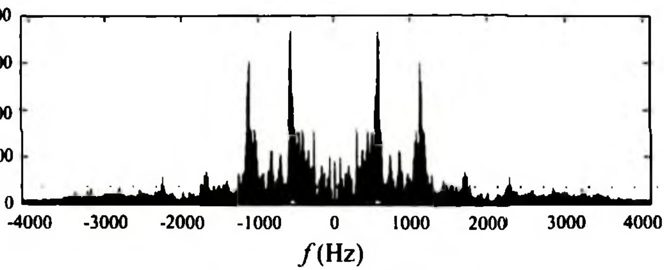
```
clear;
load handel;
xt = y;
ns = length(y);
dt = 1/Fs;                  % Time sampling interval
df = 1/((ns-1)*dt);         % Frequency sampling interval
ta = (0:dt:(ns-1)*dt)';     % Time axis
fa = (0:df:(ns-1)*df)';     % Frequency axis

figure(2);
subplot(3,2,1); plot(ta,xt);
xlabel('t (s)');
```

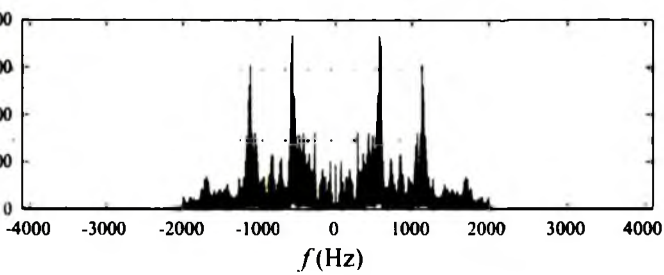
```
xf = fft(xt);

subplot(3,2,2); plot(fa,abs(xf));
xlabel('f (Hz)'); axis([0 8192 0 800]);

xf = fftshift(xf);          % Shift frequency domain data
if mod(ns,2) == 0           % Shift frequency axis
    fa = fa - ns/2*df;
else
    fa = fa - (ns-1)/2*df;
end
subplot(3,2,3); plot(fa,abs(xf));
xlabel('f (Hz)'); axis([-4096 4096 0 800]);

for id = 1:ns               % Abandon high frequency
    if (abs(fa(id)) > 2000)
        xf(id) = 0;
    end
end
subplot(3,2,4); plot(fa,abs(xf));
xlabel('f (Hz)'); axis([-4096 4096 0 800]);

xf = ifftshift(xf);         % Shift frequency domain data
if mod(ns,2) == 0           % Shift frequency axis
    fa = fa + ns/2*df;
else
    fa = fa + (ns-1)/2*df;
end
subplot(3,2,5); plot(fa,abs(xf));
xlabel('f (Hz)'); axis([0 8192 0 800]);

xt = ifft(xf);

subplot(3,2,6); plot(ta,xt);
xlabel('t (s)');
```

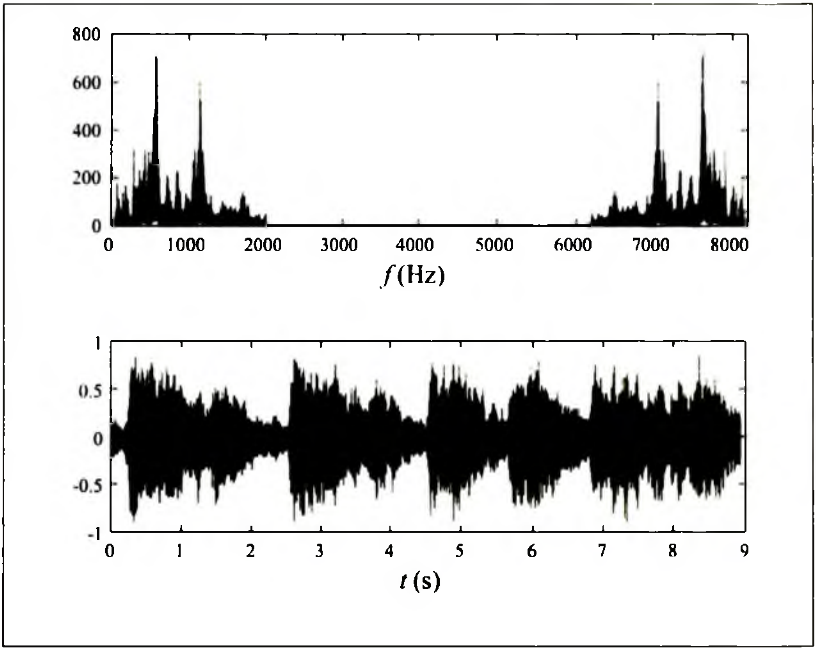
Solution


1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
7
8
9
t (s)


800
600
400
200
0
0
1000
2000
3000
4000
5000
6000
7000
8000
f (Hz)


800
600
400
200
0
-4000
-3000
-2000
-1000
0
1000
2000
3000
4000
f (Hz)


800
600
400
200
0
-4000
-3000
-2000
-1000
0
1000
2000
3000
4000
f (Hz)



Example 12.2 demonstrates how one can edit the frequency content of a time sequence. We first perform the fast Fourier transform of the time sequence "Hallelujah" via the MATLAB function `fft`. We then shift the frequency domain data via the MATLAB function `fftshift` and shift the frequency axis as follows:

$$0 \leq f \leq 2f_{\max} \quad \longrightarrow \quad -f_{\max} \leq f \leq f_{\max}.$$

Note that we need to consider two cases: one for even-numbered N and the other for odd-numbered N. In case of shifting the frequency domain data, `fftshift` automatically takes care of the two cases.

Once we have adjusted the data and axis, we may perform whatever processing we want. In Example 12.2, we are removing frequency components higher than 2000 Hz. While performing frequency domain processing, however, we should not violate the symmetry requirement in the frequency domain that amplitude and phase spectra should be even and odd functions of frequency, respectively. Losing the frequency domain symmetry makes one to face complex-valued time sequence upon the inverse transform.

Finishing the frequency domain processing, we shift back the frequency domain data via the MATLAB function ifftshift. And we perform the inverse transform via the MATLAB function ifft. The last subplot of Example 12.2 shows the time sequence "Hallelujah" that has lost its high frequency components. One may verify the result of frequency modification by entering the following commands within the MATLAB Command Window:

```
soundsc(y,Fs);
soundsc(xt,Fs);
```

The first command plays "Hallelujah" with the original frequency components, while the second plays the one without high frequency.

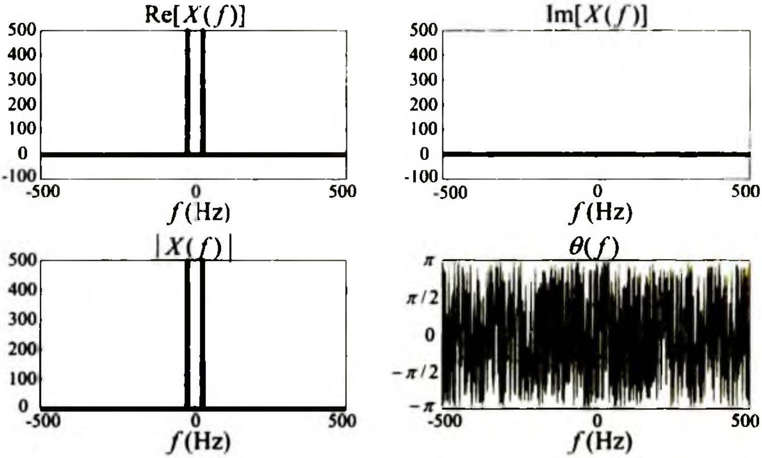
12.2.2 FFT Exercise with Sinusoidal Functions

Example 12.3 Use the following MATLAB script and plot the real part, imaginary part, amplitude spectrum, and phase spectrum of $x(t) = \cos(2\pi f_0 t)$ with $f_0 = 25$ Hz.

```
clear;
fs = 1000;
ns = 1000;
dt = 1/fs;
df = 1/((ns-1)*dt);
ta = (0:dt:(ns-1)*dt)';
fa = (0:df:(ns-1)*df)';

xt = cos(2*pi*25*ta);
if mod(ns,2) == 0
    fa = fa - ns/2*df;
else
    fa = fa - (ns-1)/2*df;
end
xf = fft(xt);
xf = fftshift(xf);
xf_re = real(xf);
```

```
xf_im = imag(xf);
xf_ab = abs(xf);
xf_ph = angle(xf);
```

Solution



We have studied in Chapter 9 that the Fourier transform of cosine functions is described as

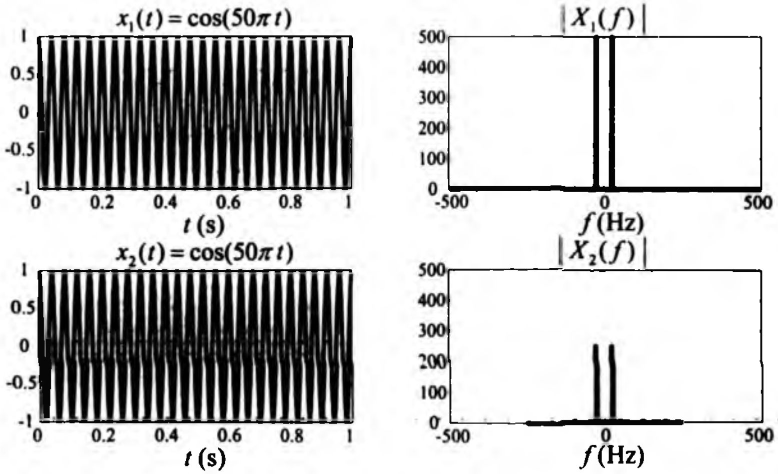
$$\cos(2\pi f_0 t) = \pi\left[\delta(f+f_0)+\delta(f-f_0)\right].$$

Example 12.3 demonstrates how the discrete Fourier transform mimics the Fourier transform of the cosine function. It is evident that the real and imaginary parts of the DFT successfully imitate the analytic expression. The phase spectrum, on the other hand, shows a noisy pattern. The reason for the noisy phase spectrum is because the absolute values of the real part data are too small. Recall the definition of the phase angle of a complex number (expression C.12). It is obvious from the definition that for complex numbers whose real parts are relatively small, the phase values may vary arbitrarily. In other words, the noisy pattern of phase spectrum in Example 12.3 is nothing more than numerical noise. The phase spectrum exemplifies that we occasionally need to remove the numerical noise via additional scripts. And, in case of Example 12.3, we can use the following script:

```
for id = 1:ns
    if xf_ab(id) < 0.1
        xf_ph(id) = 0.0;
    end
end
```

Example 12.4 Plot $x(t)$ and $|X(f)|$ of Example 12.3. Change values of both `fs` and `ns` to 500, and plot $x(t)$ and $|X(f)|$ once again.

Solution



Example 12.4 exhibits the cosine function of Example 12.3 along with its amplitude spectrum. We consider two different cases: $x_1(t)$ and $|X_1(f)|$ are plotted with

$$f_s = 1000 \text{ Hz} \quad \text{and} \quad N = 1000,$$

while $x_2(t)$ and $|X_2(f)|$ are generated with

$$f_s = 500 \text{ Hz} \quad \text{and} \quad N = 500.$$

The two time functions, $x_1(t)$ and $x_2(t)$, show almost identical graphs for 1 second. In case of amplitude spectra, $|X_1(f)|$ and $|X_2(f)|$ exhibit two amplitude peaks at identical frequencies but with different heights. In fact, the height of the impulsive peaks are given as $N/2$, in this specific case. Generally speaking, the heights of peaks in amplitude spectra are dependent on sample numbers. And, within an amplitude spectrum, an amplitude value itself does not provide any significant meaning. What is significant within an amplitude spectrum is the relative variation of amplitude.

12.3 FILTERING

We have introduced in Chapter 10 the concept of *filters* and discussed amplitude spectra of several filter types. We have also argued that the RC and RL circuits can be used as analog filters. Utilizing the FFT, we can design digital filters. A good example of digital filtering is in Example 12.2, where we remove frequency components higher than 2000 Hz. In other words, the frequency domain processing demonstrated in Example 12.2 is equivalent to the low pass filtering.

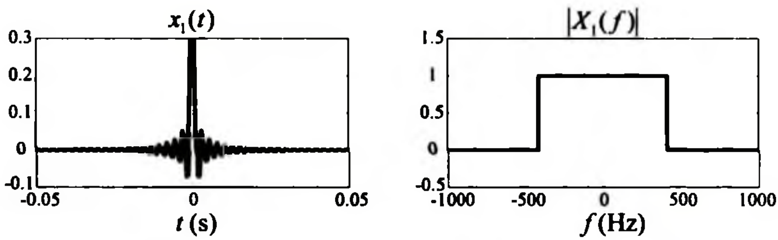


Figure 12.4: An example of low pass filter

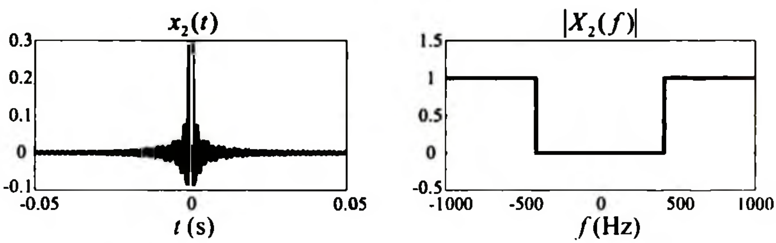


Figure 12.5: An example of high pass filter

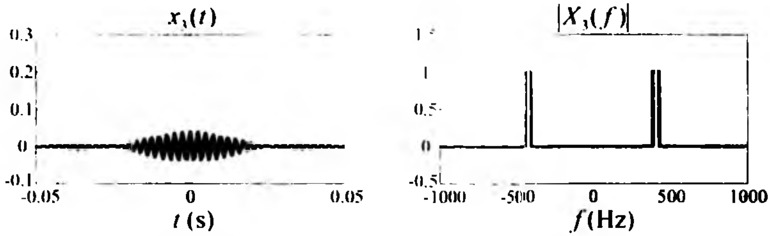


Figure 12.6: An example of band pass filter

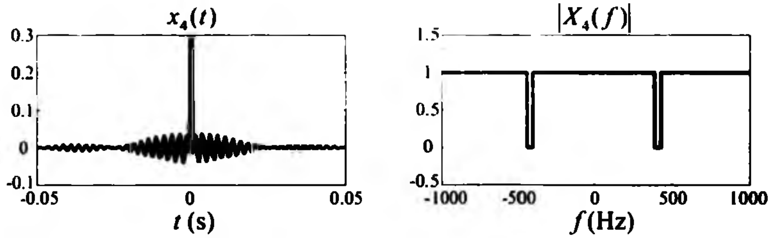


Figure 12.7: An example of band stop filter

Figure 12.4 exemplifies amplitude spectra of *low pass filters*. Time domain representation of the low pass filter is also depicted in Figure 12.4. The cutoff frequency may differ for each different filter, but Figure 12.4 well demonstrates the nature of low pass filters in time and frequency domains. Figure 12.5 illustrates the nature of another type of filters: *high pass filters*. Examples of *band pass filters* and *band stop filters* are also presented in Figures 12.6 and 12.7. Note that filters in Figures 12.4 through 12.7 are related as follows:

$$x_2(t) = \delta(t) - x_1(t) \quad \text{and} \quad X_2(f) = 1 - X_1(f),$$
$$x_4(t) = \delta(t) - x_3(t) \quad \text{and} \quad X_4(f) = 1 - X_3(f).$$

Example 12.5 Consider the following function:

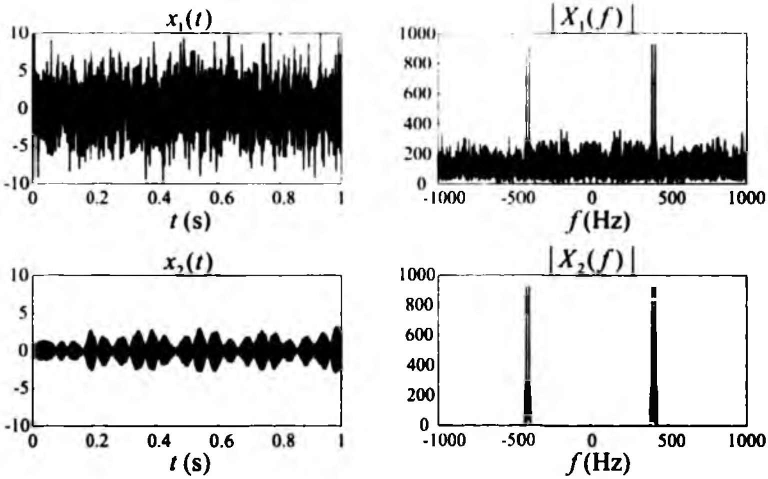
$$x_0(t) = \sin(2\pi f_1 t) + \cos(2\pi f_2 t),$$

with f_1 and f_2 are 400 Hz and 420 Hz, respectively. Sample the above function for a second with the sampling frequency $f_s = 2000$ Hz. Add random noise to the sampled data. The mean and standard deviation of the random noise should be 0 and 3, respectively. We now call the noisy data $x_1(t)$. Reduce the noise level of $x_1(t)$ via using a BPF that passes frequency components between 390 and 430 Hz. We call the noise-reduced data $x_2(t)$. Plot $x_1(t)$, $|X_1(f)|$, $x_2(t)$, and $|X_2(f)|$.

Solution

```
clear;
fs = 2000;
ns = 2000;
dt = 1/fs;
df = 1/((ns-1)*dt);
ta = (0:dt:(ns-1)*dt)';
fa = (0:df:(ns-1)*df)';
if mod(ns,2) == 0
    fa = fa - ns/2*df;
else
    fa = fa - (ns-1)/2*df;
end

xt0 = sin(2*pi*400*ta)+cos(2*pi*420*ta);
xt1 = xt0 + 3*randn(ns,1);
xf1 = fftshift(fft(xt1));

bpf = zeros(ns,1);
for id = 1:ns
    if ((abs(fa(id)) > 390) && (abs(fa(id)) < 430))
        bpf(id) = 1.0;
    end
end
xf2 = xf1.*bpf;
xt2 = ifft(ifftshift(xf2));

figure(5);
subplot(2,2,1); plot(ta,xt1);
xlabel('t (s)'); axis([0 1 -10 10]);
subplot(2,2,2); plot(fa,abs(xf1));
xlabel('f (Hz)'); axis([-1000 1000 0 1000]);
```

```
subplot(2,2,3); plot(ta,xt2);
xlabel('t (s)'); axis([0 1 -10 10]);
subplot(2,2,4); plot(fa,abs(fftshift(xf2)));
xlabel('f (Hz)'); axis([-1000 1000 0 1000]);
```



Finishing Example 12.5, readers are encouraged to check the effect of the noise reduction by the following command:

```
soundsc(xt0,fs);
soundsc(xt1,fs);
soundsc(xt2,fs);
```

12.4 WINDOWING

Sampling a signal means that we truncate sampling at one moment and limit the signal to a finite length. This truncation usually accompanies abrupt changes at either end of signals and may cause numerical artifacts called the *spectral leakage*. Consider, for example, the time signal $x(t)$ shown in Figure 12.8. The time signal is a sinusoidal signal that contains only one frequency component, and its amplitude spectrum should

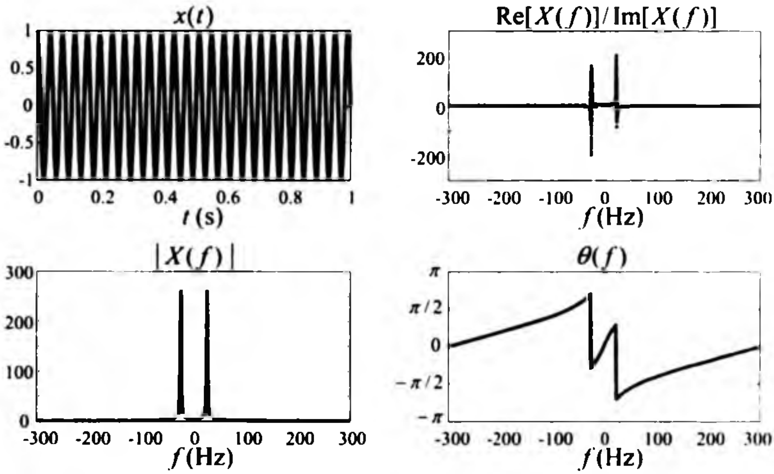


Figure 12.8: An example of spectral leakage

only exhibit two impulsive peaks. Unlike the spectrum in Example 12.4, however, the amplitude spectrum in Figure 12.8 shows that the signal contains a range of frequency components. Moreover, the phase spectrum even presents an interesting trend. This anomalous spectral pattern is the spectral leakage that arises from the sudden beginning or termination of sampling.

To reduce the spectral leakage, one may multiply data samples with windows that enforces smooth variation at the start and end of a sampling:

$$\tilde{x}[n] = x[n]\, w[n].$$

Countless *windowing* sequences have been implemented, and several representative ones are as follows:

Hann window: $w[n] = 0.5 - 0.5\cos\left(\dfrac{2\pi n}{N-1}\right)$,

Hamming window: $w[n] = 0.54 - 0.46\cos\left(\dfrac{2\pi n}{N-1}\right)$,

Blackman window: $w[n] = 0.42 - 0.5\cos\left(\dfrac{2\pi n}{N-1}\right) + 0.08\cos\left(\dfrac{4\pi n}{N-1}\right)$,

Gaussian window: $w[n] = \exp\left(-\dfrac{(2n-N-1)^2}{2\sigma^2(N-1)^2}\right)$ with $(\sigma \le 0.5)$.

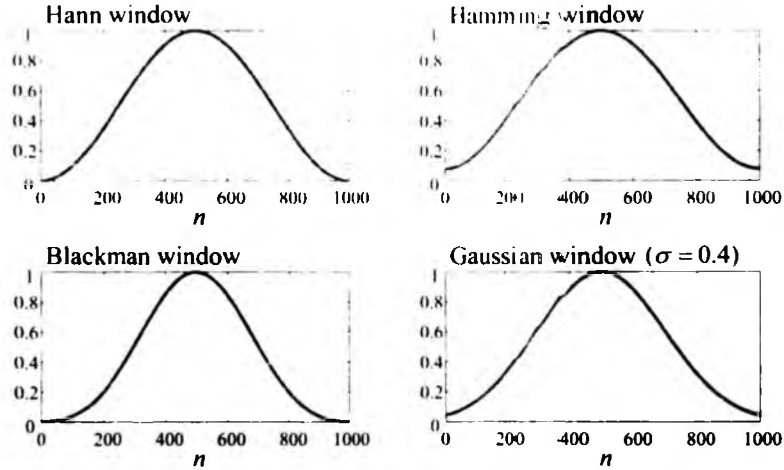


Figure 12.9: Plots of several representative windows

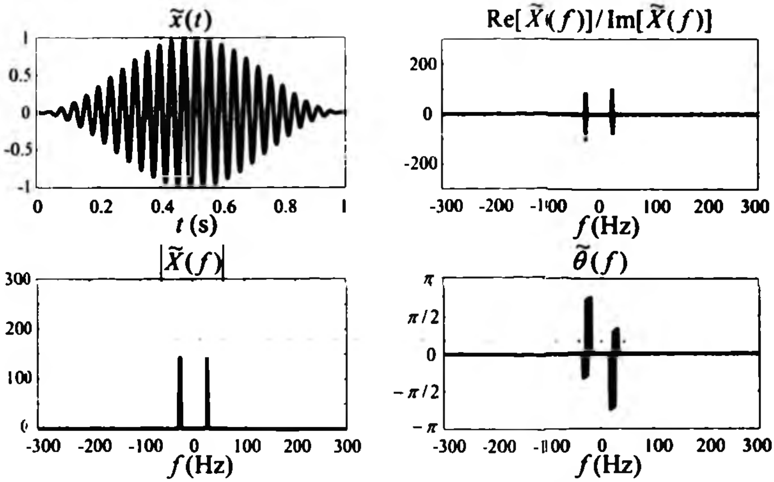


Figure 12.10: An example of windowing. Spectral leakage demonstrated in Figure 12.8 is reduced by using Hann window.

for $n = 0, 1, 2, \cdots N-1$. Figure 12.9 shows the graphs of those windows, and Figure 12.10 demonstrates that one can significantly reduce the spectral leakage shown in Figure 12.8 with the use of Hann window.

PROBLEMS

Problem 12.1 Suppose you acquire a time series data for 20 seconds with a sampling interval 5 ms. What is the frequency range that your data can effectively represent?

Problem 12.2 Suppose you acquire a time series data for 10 seconds with a sampling interval 2 ms. What is the frequency range that your data can effectively represent?

Problem 12.3 You want to measure signals whose target frequency range is between 20 kHz and 5 MHz. What is the maximum allowed sampling interval? And what is the minimum required sampling duration?

Problem 12.4 You want to measure signals whose target frequency range is between 5 kHz and 1 MHz. What is the maximum allowed sampling interval? And what is the minimum required sampling duration?

Chapter 13

BROADER PERSPECTIVE

13.1 DIFFERENT FORMS OF FOURIER TRANSFORM

Before introducing another topic of Fourier analysis, it is necessary to point out that there exist different versions of Fourier transform. Below are examples of Fourier transform pairs that are equally valid with each other.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \quad \Leftrightarrow \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \quad \Leftrightarrow \quad X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \quad \Leftrightarrow \quad X(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega \quad \Leftrightarrow \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt$$

Note that the location of the factor $1/2\pi$ does not matter. Note also that $e^{j\omega t}$ and $e^{-j\omega t}$ may interchange their locations. The first Fourier transform pair is the one we are using in this study.

Depending on which discipline we belong to, we may use one of many different forms of Fourier transform pairs. Furthermore, depending on the form of the Fourier transform pair, most of the mathematical discussion covered in this study should be expressed in a different way. As a result of that, it is important to remember that any equation or experiment result we intend to use might be based on a Fourier transform pair different from the one we are using.

13.2 TWO-DIMENSIONAL FOURIER TRANSFORM

As was mentioned in Chapter 1, there can be multidimensional signals. A good example of multidimensional signal processing is the image processing (Gonzalez and Woods 2018). In this study, we only consider processing static images that do not vary with time. In other words, we consider image signals that only vary with space. Additionally, we suggest making clear distinctions between time and spatial signals as follows: time signals are Fourier transformed to frequency domain, while spatial signals are transformed to *wavenumber domain*.

Recall that angular frequency ω is defined as

$$\omega = \frac{2\pi}{T}, \tag{13.1}$$

where T denotes the period of a time signal. Similarly, *wavenumber* k is defined as

$$k = \frac{2\pi}{\lambda}, \tag{13.2}$$

where λ denotes the wavelength of a spatial signal. Fourier transform of a spatial signal $i(x)$ can thus be expressed as

$$I(k) = \int_{-\infty}^{\infty} i(x)\, e^{-jkx}\, dx, \tag{13.3}$$

and its inverse Fourier transform as

$$i(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I(k)\, e^{jkx}\, dk. \tag{13.4}$$

Images are 2-dimensional signals that vary along two spatial axes, and we extend the above expressions as

$$I(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i(x, y)\, e^{-j(k_x x + k_y y)}\, dx\, dy, \tag{13.5}$$

and

$$i(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(k_x, k_y)\, e^{j(k_x x + k_y y)}\, dk_x\, dk_y, \tag{13.6}$$

where k_x and k_y are x-directional and y-directional wavenumbers, respectively. Expressions 13.5 and 13.6 provide mathematical prototypes of 2-dimensional Fourier transform.

Discrete Fourier transform (DFT) between space and wavenumber domain is not different from the one between time and frequency domain. And we can readily express a discrete Fourier transform pair as

$$I[m] = \sum_{n=0}^{N-1} i[n]\, e^{-j2\pi mn/N} \quad \text{and} \quad i[n] = \frac{1}{N}\sum_{m=0}^{N-1} I[m]\, e^{j2\pi mn/N}.$$

Extending DFT into 2-dimension is also straightforward such that

$$I[m_x, m_y] = \sum_{n_y=0}^{N_y-1} \sum_{n_x=0}^{N_x-1} i[n_x, n_y]\, e^{-j2\pi(m_x n_x/N_x + m_y n_y/N_y)}, \tag{13.7}$$

and its inverse discrete Fourier transform is

$$i[n_x, n_y] = \frac{1}{N_x N_y} \sum_{m_y=0}^{N_y-1} \sum_{m_x=0}^{N_x-1} I[m_x, m_y]\, e^{j2\pi(m_x n_x/N_x + m_y n_y/N_y)}, \tag{13.8}$$

where N_x and N_y represent the x-directional and y-directional lengths of a 2-dimensional array, respectively.

We have discussed in Chapter 11 that DFT is defined for periodic sequences. Same is true for 2-dimensional problems. In other words,

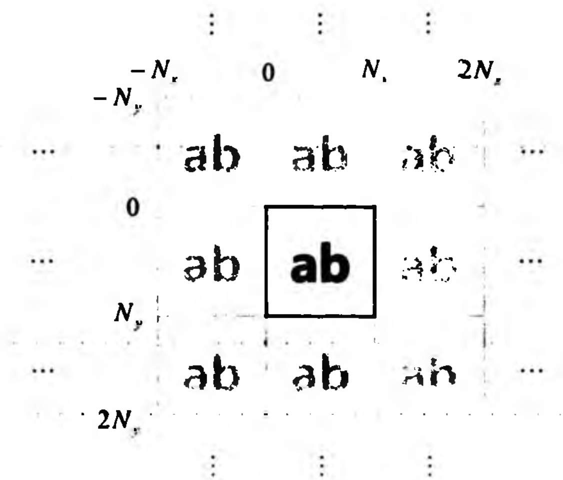


Figure 13.1: Periodicity related to 2-dimensional discrete Fourier transforms

expressions 13.7 and 13.8 are defined for 2-dimensional periodic arrays. Most of the 2-dimensional images that we handle are, however, not periodic. We should therefore keep in mind that DFT algorithms always regard nonperiodic images that we provide as periodic arrays. DFT algorithms also regard the 2-dimensional pixel numbers N_x and N_y of a 2-dimensional image as the x-directional and y-directional wavelengths, respectively. Figure 13.1 depicts the periodicity related to the 2-dimensional DFT.

A 2-dimensional discrete Fourier transform is composed of a series of 1-dimensional transforms. Consider, for example, the 2-dimensional array shown in Figure 13.2 (a). The size of the 2-dimensional array is N_x by N_y along the x and y axes directions, respectively. Among the two axes directions, one may arbitrarily choose a direction and start performing 1-dimensional DFTs. Choosing y-axes direction first, as illustrated in Figure 13.2 (b), one needs to perform N_x times of y-directional DFTs. And then, one switches direction and performs N_y times of x-directional DFTs. In other words, $N_x + N_y$ times of 1-dimensional DFTs in total constitute a single 2-dimensional transform.

While using MATLAB one does not need to perform a large number of 1-dimensional discrete Fourier transforms, because MATLAB function `fft2` automates aforementioned processes to achieve a 2-dimensional DFT. Likewise, inverse DFT of a 2-dimensional array can be easily achieved by the MATLAB function `ifft2`. There are, however, things that demand our special our attention. We have studied with Examples 12.2 and 12.5 that it is preferable to perform frequency domain shifting right after a DFT or before an inverse DFT. With regard to a 2-dimensional image processing, we also suggest to shift wavenumber domain data immediately after the 2-dimensional DFT. We then proceed with signal processing

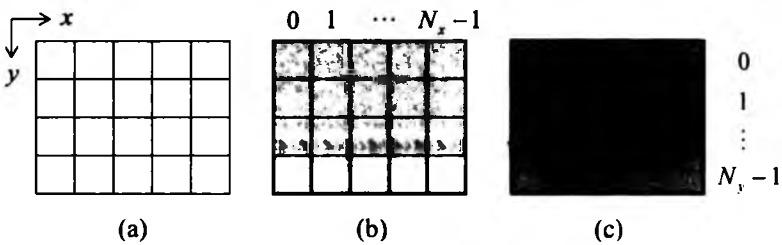


Figure 13.2: 2-dimensional discrete Fourier transform as a combination of 1-dimensional transforms

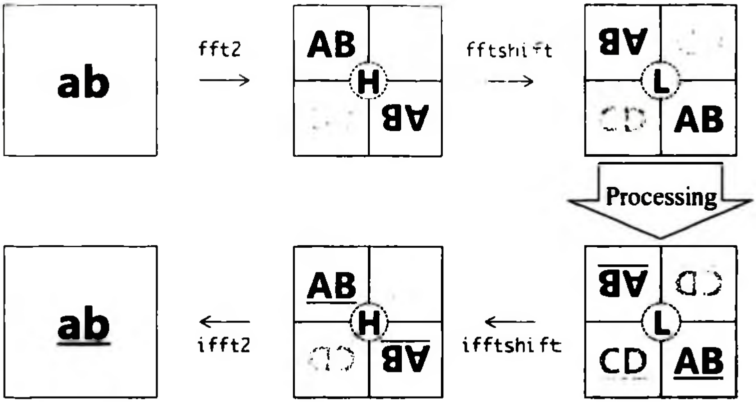


Figure 13.3: 2-dimensional fast Fourier transform via MATLAB. Lower case letters represent space domain sequences, while upper case letter represent wavenumber domain sequences.

within the wavenumber domain and back-shift the wavenumber domain data immediately before the inverse DFT.

The procedure of a 2-dimensional shifting is illustrated in Figure 13.3. 2-dimensional DFTs arrange wavenumber domain data in such a way that the highest wavenumber components locate at the center of the 2-dimensional arrays. Similarly to the 1-dimensional Fourier analysis, which we have studied in Chapter 12, it is preferable to rearrange the wavenumber domain arrays so that zero wavenumber components situate at the center of the 2-dimensional arrays. The MATLAB function `fftshift` does the rearrangement and allows one to safely perform the rest of wavenumber domain processing. Note that within the wavenumber domain, any two points on the opposite side of the center have identical amplitude but opposite sign of phase value. Therefore, while performing wavenumber domain processing, one has to keep the central symmetry of the 2-dimensional arrays. Losing the central symmetry yields a space domain array that is full of complex numbers.

13.3 FIRST LADY OF THE INTERNET

We demonstrate 2-dimensional DFT via one of the most famous images within the image processing community. Figure 13.4 shows the image called "Lenna". Lenna has been a standard test image in the field of image

Figure 13.4: The First Lady of the Internet

processing since 1973. And the lady within the image is known to be the "First Lady of the Internet". The amplitude and phase spectra of Lenna is shown in Figure 13.5. Note that the wavenumber domain data are already rearranged such that the center of the spectral images correspond to zero wavenumber. Note also that the spectral images do exhibit central symmetry.

One may wonder what the significance of phase spectra are, especially in comparison to amplitude spectra. One may further regard phase spectrum less significant than amplitude spectra. It is true that most of the time, we take a look at amplitude spectra instead of phase spectra. However, that does not necessarily mean phase information is less significant than amplitude information. Figure 13.6 shows an interesting test result. We at first keep both the amplitude and phase information of Lenna and do the inverse DFT. The inverse DFT, of course, yields the original image.

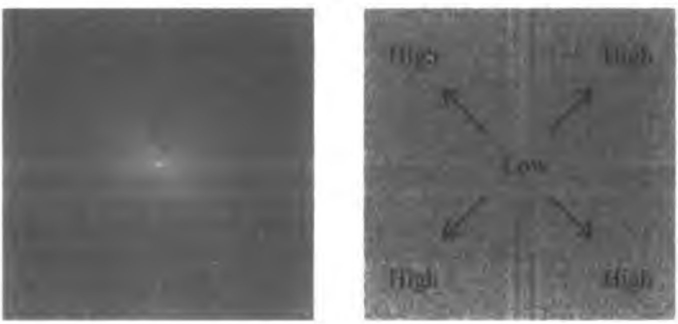


Figure 13.5: Amplitude and phase spectra of Lenna

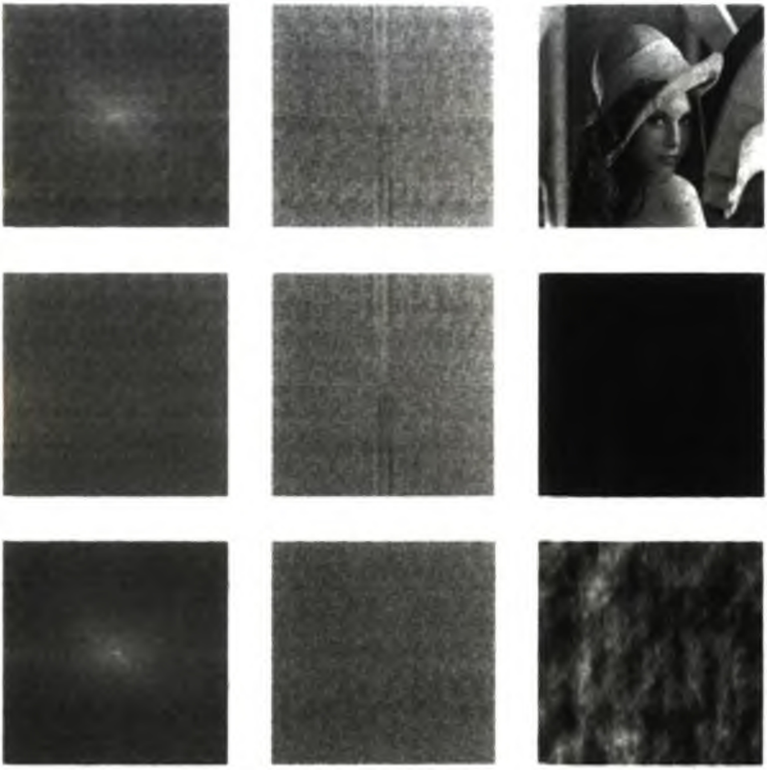


Figure 13.6: Significance of the amplitude and phase spectra illustrated via Lenna

Secondly, we replace the amplitude information with a random noise but keep the phase spectrum as it is. Surprisingly, the inverse DFT produces an image that still preserves the outline of the First Lady of the Internet. Finally, we preserve the amplitude information but destroy the phase information. The inverse DFT now reconstructs an image that is unable for one to identify the content of it. Figure 13.6 thus demonstrates that although we mainly use amplitude spectra to represent certain procedures of wavenumber domain processing, we should never ignore the significance of phase information.

Figure 13.7 shows examples of 2-dimensional low pass filtering. The amplitude spectra in the upper row qualitatively illustrate the extent of high wavenumber components that are being filtered out. Underneath each

Figure 13.7: Low pass filtering of Lenna

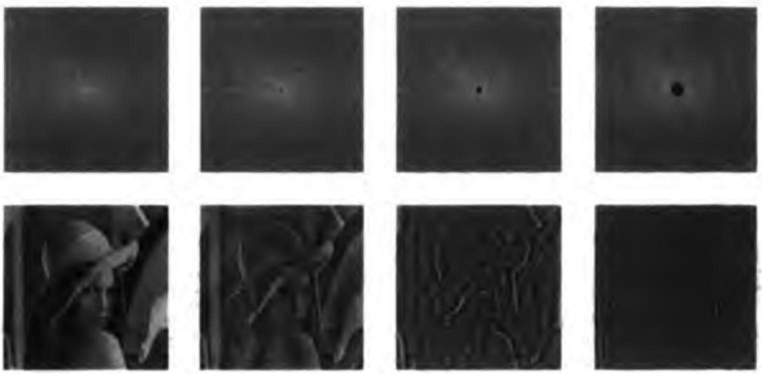


Figure 13.8: High pass filtering of Lenna

different amplitude spectra are images that come from the inverse DFT of the low pass filtered wavenumber domain data. It is obvious that the more high wavenumber components filtered out, the more blurry image we get. Figure 13.8, on the other hand, exemplifies high pass filtering. It is evident that the more low wavenumber components removed, the sharper image the inverse DFT yields. It is also noteworthy that high pass filtering can be useful for detecting edges of an object.

Filtering out certain wavenumber components is not the only thing one can achieve in the wavenumber domain. One may also reinforce certain wavenumber components. Figures 13.9 and 13.10 illustrate what Lenna would look like if one amplifies low wavenumber and high wavenum-

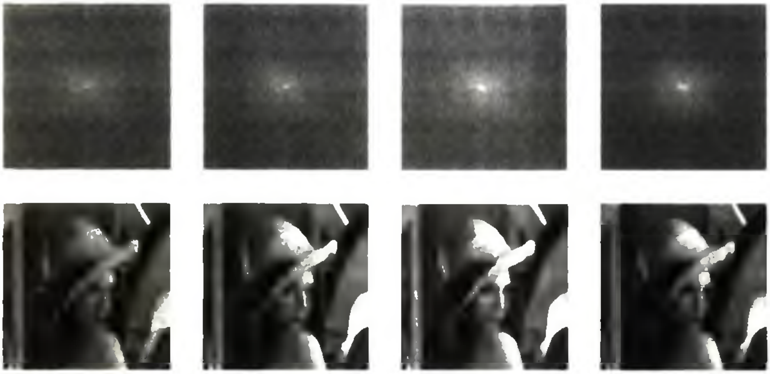


Figure 13.9: Amplifying low wavenumber components of Lenna

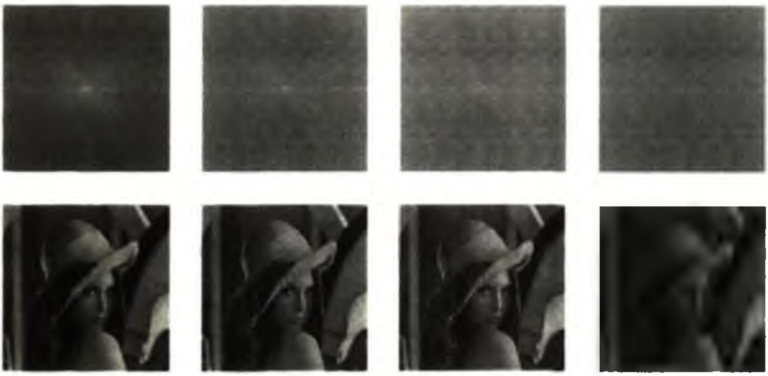


Figure 13.10: Amplifying high wavenumber components of Lenna

ber components, respectively. Amplifying low wavenumber components yields soft or mild looking images, whereas amplifying high wavenumber components accompanies images that look rough and wild. Figures 13.7 - 13.10 well demonstrate that 2-dimensional DFT can be successfully utilized for digital image processing.

13.4 LAPLACE TRANSFORM AND Z-TRANSFORM

Since the beginning of Chapter 7, we have studied what Fourier analysis is about. And our discussion has been focused on Fourier transform. We should now introduce another integral transform that is widely used in science and technology. That is *Laplace transform*. We do not aim to cover details about the integral transform. Instead of that, we intend to give a brief introduction about Laplace transform and discuss the relationship between Fourier and Laplace transforms. For more details about Laplace transform, readers are encouraged to refer to other literature that include Oppenheim and Willsky (1997), Oppenheim and Schafer (2010), or Lathi and Green (2017).

Consider a time function $x(t)$ that is defined for $t > 0$. The Laplace transform $X(s)$ of the time function is described as

$$X(s) = \int_0^\infty x(t)\, e^{-st} dt. \tag{13.9}$$

And the complex parameter s is expressed as

$$s = \sigma + j\omega, \tag{13.10}$$

where σ and ω are the real and imaginary parts of s, respectively. Expression 13.9 is known as the one-sided or unilateral form of Laplace transform, because the integration range is the positive time axis. We encounter only positive time signals, and thus the unilateral form is a more commonly used transform than the two-sided or bilateral form of Laplace transform that is defined as

$$X(s) = \int_{-\infty}^\infty x(t)\, e^{-st} dt. \tag{13.11}$$

The advantage of bilateral Laplace transform is that it can handle both causal and noncausal signals over $-\infty$ to ∞.

The Laplace transform $X(s)$ is not always defined over the entire s-plane, and the region of the s-plane that $X(s)$ converges is called the *region of convergence* (ROC). As depicted in Figure 13.11, the ROC of $X(s)$ commonly consists of strips parallel to the imaginary axis in the s-plane. If $x(t)$ is of finite duration and absolutely integrable, the ROC is the entire s-plane. The *inverse Laplace transform* is also defined within the ROC and expressed as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} X(s)\, e^{st} ds. \tag{13.12}$$

Comparing expression 13.11 with expression 9.1, one can notice that Fourier and Laplace transforms are closely related. In fact, one may regard

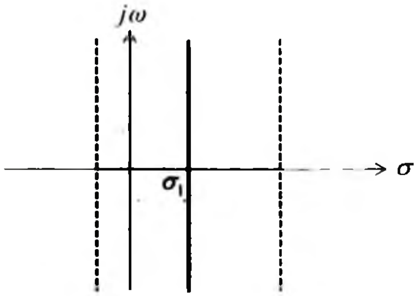


Figure 13.11: Region of convergence (ROC) of the Laplace transform

Fourier transform as a special case of Laplace transform with $\sigma = 0$ and $s = j\omega$. For circuit analysis, Laplace transform is more widely used than Fourier transform because the Laplace integral converges for a wider range of signals and automatically incorporates initial conditions.

One of many common properties between Fourier and Laplace transforms is the convolution property. Consider, for example, a linear time-invariant (LTI) system whose impulse response is $h(t)$. We denote the Laplace transform of $h(t)$ as $H(s)$ and call it the *transfer function* or system function of the LTI system. The convolution property of Laplace transform then associates the following transform pair:

$$y(t) = x(t) * h(t) \quad \Leftrightarrow \quad Y(s) = X(s)\,H(s). \tag{13.13}$$

Therefore, the transfer function can be also expressed as

$$H(s) = \frac{Y(s)}{X(s)}. \tag{13.14}$$

The transfer function is, in fact, a more generalized concept of the frequency response that we have studied in Chapters 8 and 10. Transfer function of a system describes how the output behaves with respect to the input. Given the system transfer function, one can perform extensive system analysis and design without having to apply specific signals to the system.

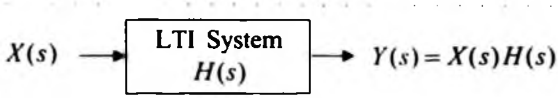


Figure 13.12: Concept of transfer function

Fourier and Laplace transforms are defined for continuous-time functions. We have discussed in Chapter 11 that one can perform Fourier analysis of a discrete-time sequence via discrete-time Fourier transform (DTFT). With regard to the discrete-time sequence, we can also perform a transform called the *z-Transform*. The relationship between Fourier transform and DTFT is similar to the one between Laplace transform and z-Transform. In other words, z-Transform is a discrete-time counterpart of Laplace transform. The z-Transform $X(z)$ of a time sequence $x[n]$ is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}, \tag{13.15}$$

where

$$z = e^{s} = e^{\sigma + j\omega}. \tag{13.16}$$

Expression 13.15 is known as the unilateral form of z-Transform. And one can also define the bilateral form of z-Transform as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}. \tag{13.17}$$

Note that with $\sigma = 0$ and $s = j\omega$, expression 13.17 reduces to expression 11.5. Note also that we have already encountered z-Transform in Chapter 6 and used it for calculating discrete-time convolutions.

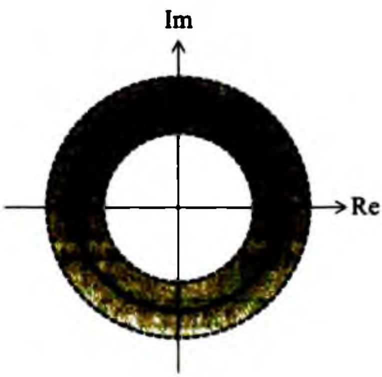


Figure 13.13: Region of convergence (ROC) of the z-Transform

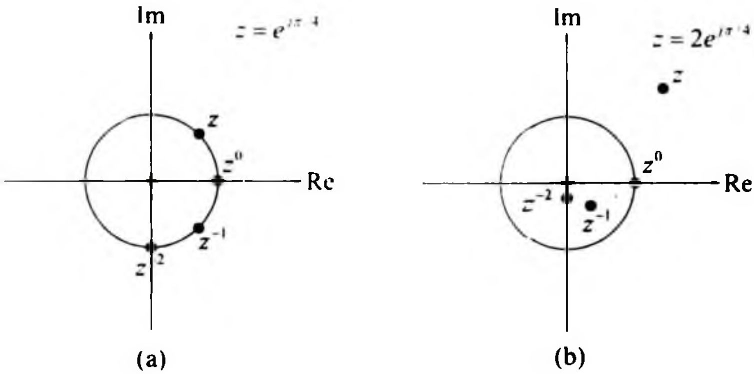


Figure 13.14: Discrete-time Fourier transform (a) vs z-Transform (b)

Similarly to Laplace transform, z-Transform is not always defined over the entire z-plane, and the region of z-plane where $X(z)$ converges is called the *region of convergence* (ROC). As depicted in Figure 13.13, the ROC of $X(z)$ commonly consists of a ring in the z-plane centered about the origin. The *inverse z-Transform* is only defined within the ROC and expressed as

$$x[n] = \frac{1}{2\pi j} \oint_{\Gamma} X(z)\, z^{n-1}\, dz. \tag{13.18}$$

Figure 13.14 highlights similarities / differences between DTFT and z-Transform. Both DTFT and z-Transform associate time sequence values with powers of a complex number. In the case of DTFT, time sequence values are associated with complex numbers on the unit circle of the z-plane. In the case of z-Transform, on the other hand, time sequence values are associated with complex numbers that stay on a spiral path of the z-plane.

Just as Laplace transform is useful in handling signals that do not have Fourier transform, z-transform enables us to treat discrete-time signals that do not have DTFT. Also, just as Laplace transform converts integro-differential equations into algebraic equations, z-transform converts difference equations into algebraic equations that are easier to manipulate and solve. Although the properties of z-transform are similar to those of Laplace transform, there are some differences. Like Laplace transform, z-transform is applicable to systems with initial conditions. z-Transform is fundamentally important to digital signal processing, digital communications, and linear control systems. Readers are strongly encouraged to refer to Oppenheim and Schafer (2010) for more detailed discussion.

Appendix A

GREEK ALPHABETS

Upper Case	Lower Case	Name
A	α	Alpha
B	β	Beta
Γ	γ	Gamma
Δ	δ	Delta
E	ϵ, ε	Epsilon
Z	ζ	Zeta
H	η	Eta
Θ	θ, ϑ	Theta
I	ι	Iota
K	$\kappa, \varkappa$	Kappa
Λ	λ	Lambda
M	μ	Mu
N	ν	Nu
Ξ	ξ	Xi
O	o	Omicron
Π	π, ϖ	Pi
P	ρ, ϱ	Rho
Σ	σ, ς	Sigma
T	τ	Tau
Υ	υ	Upsilon
Φ	ϕ, φ	Phi
X	χ	Chi
Ψ	ψ	Psi
Ω	ω	Omega

Appendix B

MATH FORMULAS

B.1 TRIGONOMETRIC IDENTITIES

$$\cos(-x) = +\cos x \tag{B.1}$$
$$\sin(-x) = -\sin x \tag{B.2}$$

$$\cos(x + \pi/2) = -\sin x \tag{B.3}$$
$$\cos(x - \pi/2) = +\sin x \tag{B.4}$$
$$\sin(x + \pi/2) = +\cos x \tag{B.5}$$
$$\sin(x - \pi/2) = -\cos x \tag{B.6}$$

$$\cos(x \pm \pi) = -\cos x \tag{B.7}$$
$$\sin(x \pm \pi) = -\sin x \tag{B.8}$$
$$\cos(x \pm 2\pi) = +\cos x \tag{B.9}$$
$$\sin(x \pm 2\pi) = +\sin x \tag{B.10}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \tag{B.11}$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y \tag{B.12}$$
$$\sin(x + y) = \sin x \cos y + \cos x \sin y \tag{B.13}$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y \tag{B.14}$$

$$2\cos x \cos y = \cos(x + y) + \cos(x - y) \tag{B.15}$$
$$2\sin x \sin y = \cos(x - y) - \cos(x + y) \tag{B.16}$$
$$2\sin x \cos y = \sin(x + y) + \sin(x - y) \tag{B.17}$$
$$2\cos x \sin y = \sin(x + y) - \sin(x - y) \tag{B.18}$$

$$\cos^2 x + \sin^2 x = 1 \tag{B.19}$$

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 = 1 - 2\sin^2 x \end{aligned} \tag{B.20}$$

$$\sin(2x) = 2\sin x \cos x \tag{B.21}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \tag{B.22}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \tag{B.23}$$

$$e^{jx} = \cos x + j\sin x \tag{B.24}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \tag{B.25}$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \tag{B.26}$$

B.2 EXPONENTIAL / LOGARITHMIC IDENTITIES

$$e^x e^y = e^{x+y} \tag{B.27}$$

$$e^x / e^y = e^{x-y} \tag{B.28}$$

$$(e^x)^y = e^{xy} \tag{B.29}$$

$$\ln(xy) = \ln x + \ln y \tag{B.30}$$

$$\ln(x/y) = \ln x - \ln y \tag{B.31}$$

$$\ln(x^y) = y\ln x \tag{B.32}$$

$$e^{\ln x} = x \tag{B.33}$$

$$\ln(e^x) = x \tag{B.34}$$

B.3 INDEFINITE INTEGRALS

$$\int a\,dx = ax + C \tag{B.35}$$

$$\int ax^n\,dx = \frac{a}{n+1}x^{n+1} + C \qquad \text{for} \quad n \neq -1 \tag{B.36}$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C \tag{B.37}$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C \tag{B.38}$$

$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax) + C \tag{B.39}$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) + C \tag{B.40}$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C \tag{B.41}$$

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) + C \tag{B.42}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C \tag{B.43}$$

$$\int xe^{ax}\,dx = \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax} + C \tag{B.44}$$

$$\int \frac{a}{x}\,dx = a\ln x + C \tag{B.45}$$

$$\int \ln(ax)\,dx = x\ln(ax) - x + C \tag{B.46}$$

B.4 SUMS

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{B.47}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{B.48}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

$$= (1 + 2 + 3 + \cdots + n)^2 = \left|\frac{n(n+1)}{2}\right|^2 \tag{B.49}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \tag{B.50}$$

$$\sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n$$

$$= \frac{a(1 - r^{n+1})}{1 - r} = \frac{a(r^{n+1} - 1)}{r - 1} \tag{B.51}$$

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots$$

$$= \frac{a}{1 - r} \quad \text{for} \quad |r| < 1 \tag{B.52}$$

Readers are encouraged to refer to Gradshteyn and Ryzhik (2007) for more comprehensive lists of math formulas.

Appendix C

COMPLEX NUMBERS

C.1 EULER'S FORMULA

Compromising mathematical rigor, one may regard *Taylor series* as expanding an arbitrary function in the form of infinite power series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots . \tag{C.1}$$

Some functions do not have Taylor series. And, for a function that has Taylor series, the series usually converges within a finite range of x. There are, however, functions that, regardless of the value of x, always do converge. Sine, cosine, and exponential functions have this nice property.

Consider Taylor series expansion of the exponential function:

$$e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots .$$

Assigning $x = 0$ reveals that a_0 must be 1. Differentiating the above expression, we write

$$e^x = a_1 + 2\,a_2 x + 3\,a_3 x^2 + 4\,a_4 x^3 + 5\,a_4 x^4 + \cdots .$$

Assigning $x = 0$ once again shows that a_1 is 1 too. Differentiating once again, we derive

$$e^x = 2 \cdot 1\,a_2 + 3 \cdot 2\,a_3 x + 4 \cdot 3\,a_4 x^2 + 5 \cdot 4\,a_4 x^3 + 6 \cdot 5\,a_5 x^4 + \cdots .$$

It is now clear that a_2 is $1/2!$, and, repeating the process, we deduce that the Taylor series expansion of the exponential function is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots . \tag{C.2}$$

In the same manner, one can derive Taylor series expansion of cosine and sine functions as

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots , \tag{C.3}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots . \tag{C.4}$$

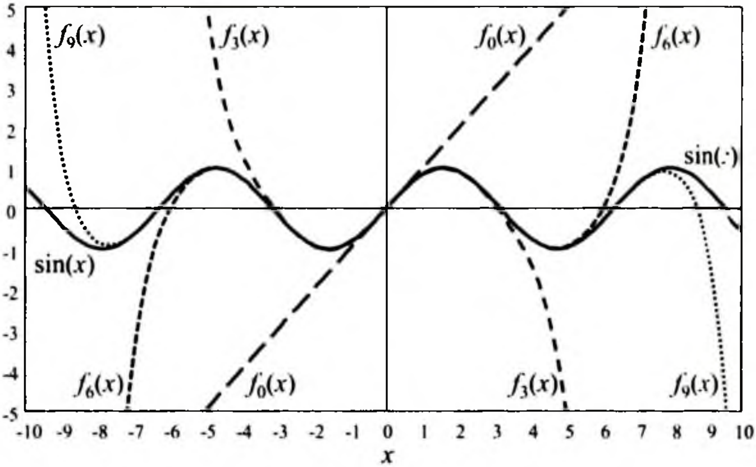


Figure C.1: Sine function vs truncated Taylor Series (expression C.5)

Figure C.1 demonstrates the convergence of expression C.4. Sine function (solid curve) is compared with the truncated Taylor series (dashed curves):

$$f_n(x) = \sum_{k=0}^{n}(-1)^k \frac{x^{2k+1}}{(2k+1)!}. \qquad (C.5)$$

It is obvious that, including higher order terms, the truncated Taylor series better approximate the sine function.

Taylor series expressions of the exponential, cosine, and sine funcions show that we are not limited to define those functions for only real values of x. In other words, Taylor series expansion enables one to define those functions for any imaginary values of x. Substituting $x = jy$, where y denotes a real variable and $j = \sqrt{-1}$, expression C.2 becomes

$$\begin{aligned} e^{jy} &= 1 + jy + \frac{(jy)^2}{2!} + \frac{(jy)^3}{3!} + \frac{(jy)^4}{4!} + \frac{(jy)^5}{5!} + \frac{(jy)^6}{6!} + \frac{(jy)^7}{7!} + \cdots \\ &= 1 + jy - \frac{y^2}{2!} - \frac{jy^3}{3!} + \frac{y^4}{4!} + \frac{jy^5}{5!} - \frac{y^6}{6!} - \frac{jy^7}{7!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + j\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots\right). \end{aligned}$$

We can therefore establish the following expression:

$$e^{jy} = \cos y + j \sin y. \qquad (C.6)$$

In honor of the great teacher of us all, expression C.6 is called the *Euler's formula*. Replacing y by $-y$ in expression C.6 gives

$$e^{-jy} = \cos y - j \sin y,$$

and combining the above two expressions yields the following useful expressions:

$$\cos y = \frac{e^{jy} + e^{-jy}}{2}, \tag{C.7}$$

$$\sin y = \frac{e^{jy} - e^{-jy}}{2j}. \tag{C.8}$$

C.2 REPRESENTATION OF COMPLEX NUMBERS

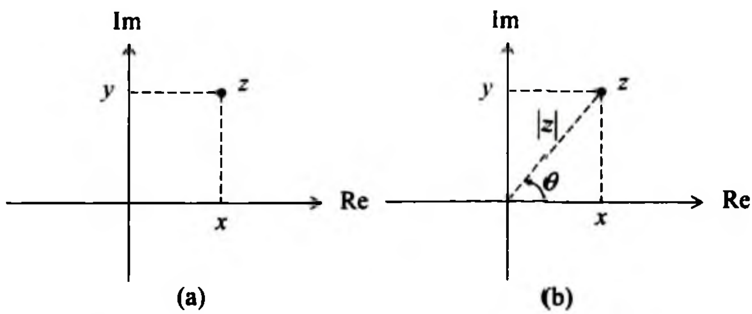


Figure C.2: Representation of a complex number: (a) Rectangular form and (b) Polar form

A complex number z is written in *rectangular form* as

$$z = x + jy, \tag{C.9}$$

where x and y are the real and imaginary parts of z, respectively (Figure C.2 (a)). Another way of representing the complex number z is called the *polar form*, and the complex number z is specified by its amplitude $|z|$ and phase angle θ (Figure C.2 (b)):

$$z = |z|\underline{/\theta}. \tag{C.10}$$

The rectangular and polar forms are associated by the following relations:

$$|z| = \sqrt{x^2 + y^2}, \tag{C.11}$$

$$\theta = \begin{cases} \tan^{-1}|y/x| & \text{(first quadrant)}, \\ \pi - \tan^{-1}|y/x| & \text{(second quadrant)}, \\ \pi + \tan^{-1}|y/x| & \text{(third quadrant)}, \\ 2\pi - \tan^{-1}|y/x| & \text{(fourth quadrant)}, \end{cases} \tag{C.12}$$

and

$$\begin{aligned} x &= |z|\cos\theta, \\ y &= |z|\sin\theta. \end{aligned}$$

Therefore, a complex number z can be also represented in the *exponential form*:

$$z = |z|\cos\theta + j|z|\sin\theta = |z|e^{j\theta}. \tag{C.13}$$

Note that *complex conjugate* (Figure C.3) of the complex number z is represented in the rectangular, polar, and exponential forms as follows:

$$z^* = \begin{cases} x - jy & \text{(rectangular form)}, \\ |z|\underline{/-\theta} & \text{(polar form)}, \\ |z|e^{-j\theta} & \text{(exponential form)}. \end{cases} \tag{C.14}$$

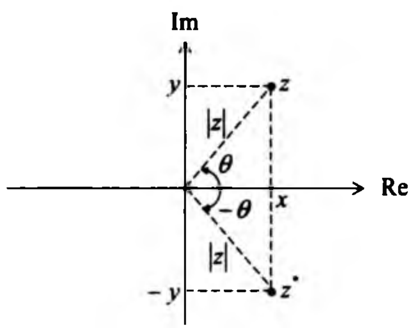


Figure C.3: Complex conjugate of a complex number

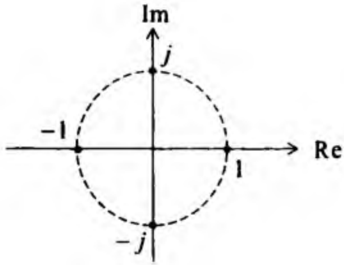


Figure C.4: Complex numbers on the unit circle

Note also that the exponential form enables one to see numbers shown in Figure C.4 with a new eye:

$$
\begin{aligned}
1 &= e^{j0} &&= e^{j2n\pi}, \\
j &= e^{j\pi/2} &&= e^{j(2n+1/2)\pi}, \\
-1 &= e^{j\pi} &&= e^{j(2n+1)\pi}, \\
-j &= e^{j3\pi/2} &&= e^{j(2n+3/2)\pi},
\end{aligned}
$$

with an integer n.

Example C.1 Express the following number in the exponential form:

$$z = \frac{1}{\sqrt{3} - j}.$$

Solution

$$z = \frac{(\sqrt{3} + j)}{(\sqrt{3} - j)(\sqrt{3} + j)} = \frac{\sqrt{3} + j}{4} = \frac{\cos(\pi/6) + j\sin(\pi/6)}{2} = \frac{1}{2}e^{j\pi/6}.$$

Example C.2 We know $2^2 = 4$ and $j^2 = -1$. What then are j^j and 2^j? Express those two numbers in the rectangular form.

Solution

$$j^j = (e^{j\pi/2})^j = e^{j^2\pi/2} = e^{-\pi/2} \approx 0.208,$$
$$2^j = e^{\ln(2^j)} = e^{j\ln(2)} = \cos(\ln(2)) + j\sin(\ln(2)) \approx 0.769 + 0.639j.$$

Note however that these solutions are not unique (Brown and Churchill 2014).

C.3 ARITHMETIC OF COMPLEX NUMBERS

With the following complex numbers:

$$z_1 = x_1 + jy_1,$$
$$z_2 = x_2 + jy_2,$$

the addition / subtraction of the two complex numbers are

$$z_1 \pm z_2 = (x_1 \pm x_2) + j(y_1 \pm y_2).$$

Multiplying the two numbers we get

$$z_1 z_2 = (x_1x_2 - y_1y_2) + j(x_2y_1 + x_1y_2),$$

and by division we have

$$\frac{z_1}{z_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

It is, however, hard to acquire geometric significance of the multiplicaion / division with the rectangular form. The exponential form, on the other hand, well delivers the geometric meaning.

Consider the following complex numbers:

$$z_1 = |z_1|e^{j\theta_1},$$
$$z_2 = |z_2|e^{j\theta_2}.$$

The multiplication / division of the two complex numbers are

$$z_1 z_2 = |z_1||z_2|e^{j(\theta_1+\theta_2)},$$
$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{j(\theta_1-\theta_2)}.$$

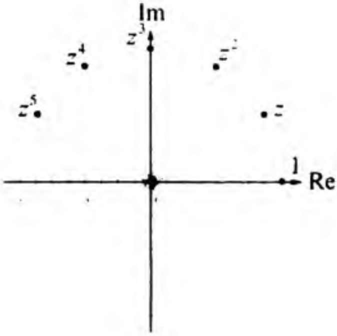


Figure C.5: Powers of a complex number

The above expressions clearly show that multiplication involves addition of angles, whereas division involves subtraction.

This interesting feature of multiplying complex numbers can be highlighted by taking power of a complex number that lies on the unit circle (Figure C.5). The complex number z is $e^{j\pi/6}$ and, thus, powers of the number are expressed in the exponential form as

$$z^n = e^{j\pi n/6}.$$

In other words, taking power of a complex number that lies on the unit circle is simply to rotate around the unit circle.

Example C.3 The number z in Figure C.5 is expressed in the rectangular form as

$$z = e^{j\pi/6} = \cos(\pi/6) + j\sin(\pi/6) = \frac{\sqrt{3}+j}{2}.$$

Calculate the other complex numbers in Figure C.5 via repeated multiplications in the rectangular form. Finishing multiplications, convert the multiplication results into the exponential form.

Solution

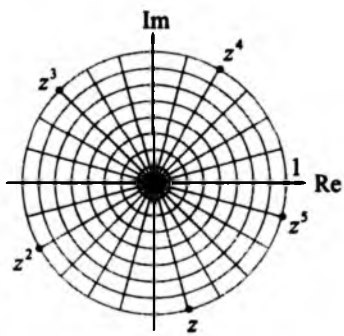
$$z^2 = z\,z = \frac{\sqrt{3}+j}{2}\,\frac{\sqrt{3}+j}{2} = \frac{2+2j\sqrt{3}}{4} = \frac{1+j\sqrt{3}}{2}$$
$$= \cos(\pi/3) + j\sin(\pi/3) = e^{j\pi/3},$$

$$z^3 = z^2\,z = \frac{1+j\sqrt{3}}{2}\,\frac{\sqrt{3}+j}{2} = \frac{4j}{4} = j = e^{j\pi/2},$$

$$z^4 = z^3\,z = j\,\frac{\sqrt{3}+j}{2} = \frac{-1+j\sqrt{3}}{2}$$
$$= \cos(2\pi/3) + j\sin(2\pi/3) = e^{j2\pi/3},$$

$$z^5 = z^4\,z = \frac{-1+j\sqrt{3}}{2}\,\frac{\sqrt{3}+j}{2} = \frac{-2\sqrt{3}+2j}{4} = \frac{-\sqrt{3}+j}{2}$$
$$= \cos(5\pi/6) + j\sin(5\pi/6) = e^{j5\pi/6}.$$

Example C.4 Consider the number z shown below. Do not use calculator and show that $z^{2020} = z^4$.



Solution

$$\begin{aligned} z &= e^{-j5\pi/12}, \\ z^{24} &= e^{-j10\pi} = (e^{-j2\pi})^5 = (1)^5 = 1, \\ z^{2020} &= z^{24\times 84+4} = (z^{24})^{84}\,z^4 = (1)^{84}\,z^4 = z^4. \end{aligned}$$

PROBLEMS

Problem C.1 Show that Taylor series expansion of the cosine function is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

Problem C.2 Show that Taylor series expansion of the sine function is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Problem C.3 Express the following number in the exponential form:

$$z = \frac{1}{1+j}.$$

Problem C.4 Express the following number in the exponential form:

$$z = \frac{2}{1+j\sqrt{3}}.$$

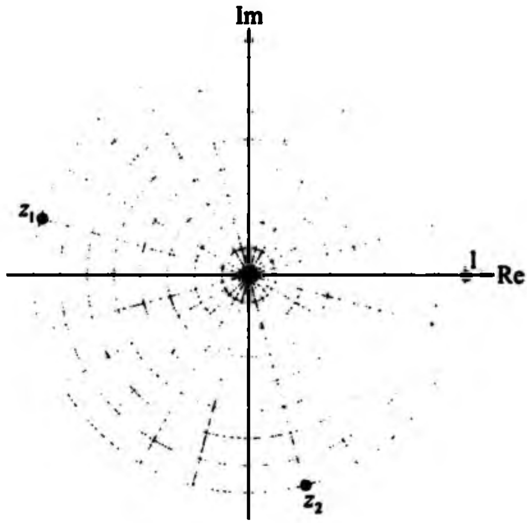
Problem C.5 Calculate z^{2020} with the following number (do not use calculator):

$$z = \frac{1}{2} + j\frac{\sqrt{3}}{2}.$$

Problem C.6 Calculate z^{2022} with the following number (do not use calculator):

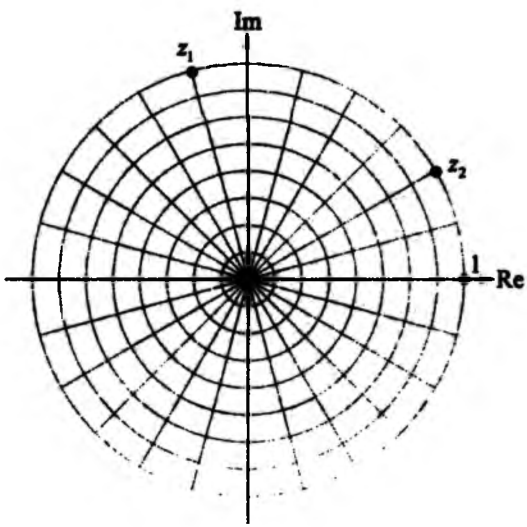
$$z = \frac{\sqrt{3}}{2} - j\frac{1}{2}.$$

Problem C.7 Consider two complex numbers shown below.



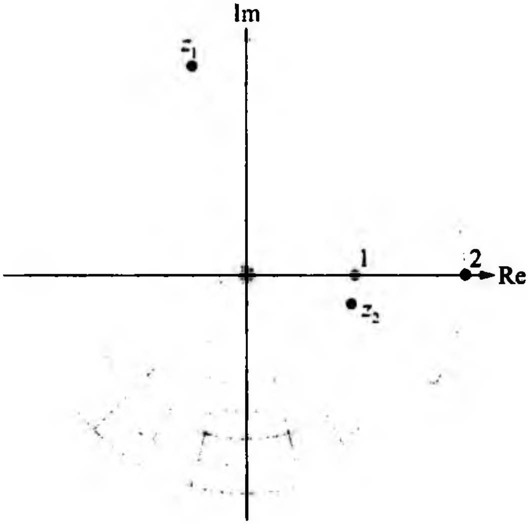
Denote locations of $z_1 z_2$, z_1/z_2, and z_2/z_1 on the figure.

Problem C.8 Consider two complex numbers shown below.



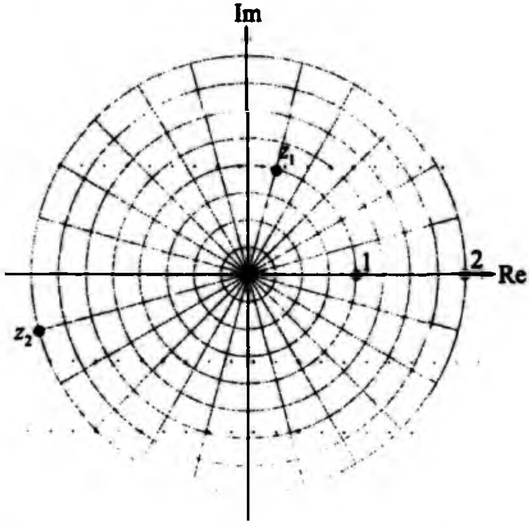
Denote locations of $z_1 z_2$, z_1/z_2, and z_2/z_1 on the figure.

Problem C.9 Consider two complex numbers shown below.



Denote locations of $z_1 z_2$, z_1/z_2, and z_2/z_1 on the figure.

Problem C.10 Consider two complex numbers shown below.



Denote locations of $z_1 z_2$, z_1/z_2, and z_2/z_1 on the figure.

Appendix D

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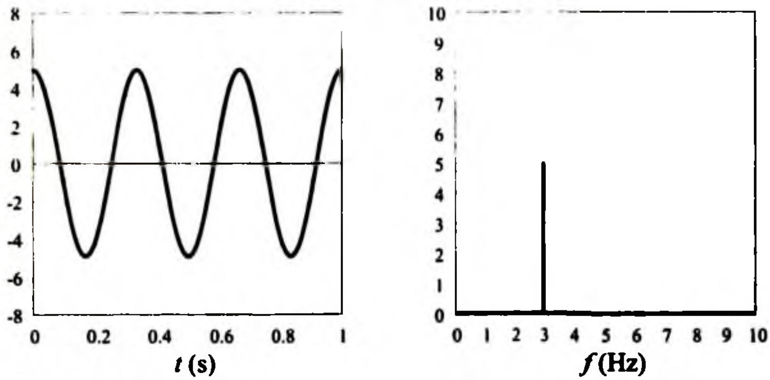
Appendix E

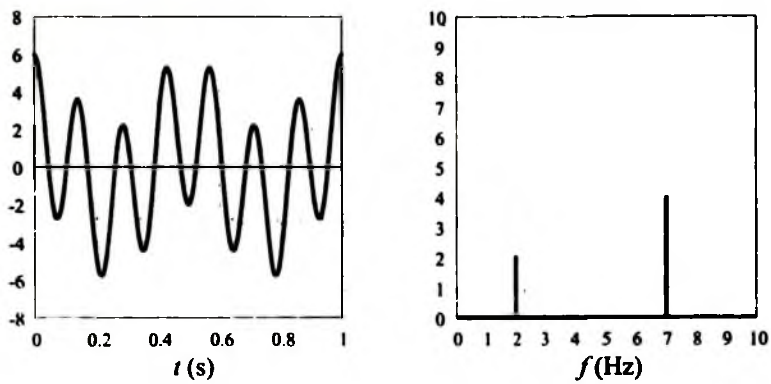
ANSWERS TO PROBLEMS

CHAPTER 1

1.1



1.2



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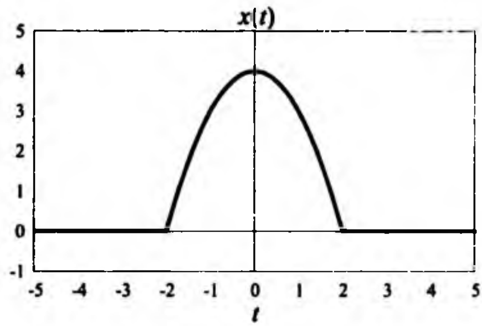
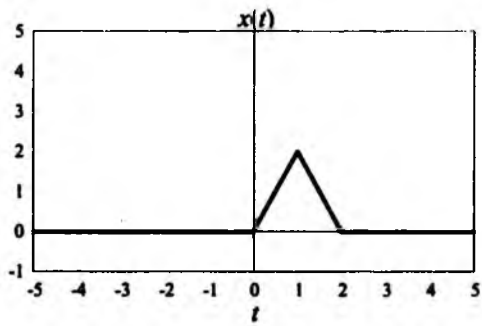


1.4



1.5 $x(t) = u(t)\,u(-t+1) + u(t-1)\,r(-t+2)$

1.6 $x(t) = 2r(t+2)\,u(-t-1) + u(t+1)\,u(-t)\,r(-t+1)$
$\quad + u(t)\,u(-t+1) + u(t-1)\,r(-t+2)$

1.7 d

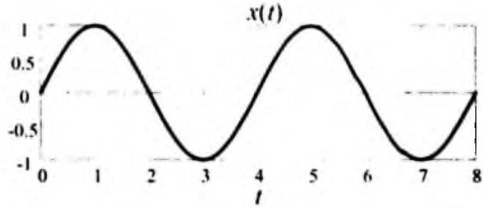
1.8 a

1.9 $x[n] = 3\delta[n+1] + 4\delta[n] + 3\delta[n-1]$

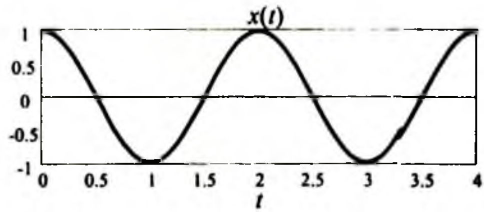
1.10 $x[n] = 4\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2]$

CHAPTER 2

2.1 $T = 4$



2.2 $T = 2$



2.3 $T_0 = 15$

2.4 $T_0 = 40$

2.5 $N_0 = 30$

2.6 $N_0 = 70$

2.7 c

2.8 d

2.9 $E = 1$. $x(t)$ is thus an energy signal.

2.10 $E = \infty$ and $P = 1$. $x(t)$ is thus a power signal.

CHAPTER 3

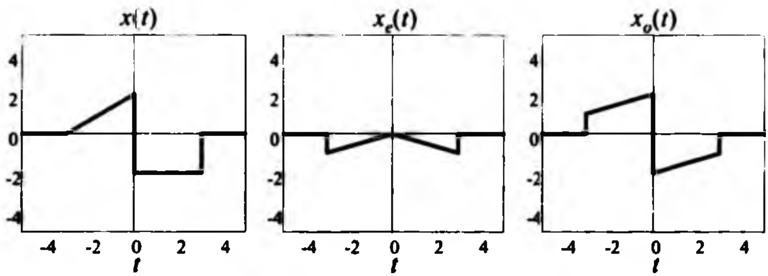
3.1 $x(t) = 4\cos(2\pi t/3 + 2\pi/3)$

3.2 $x(t) = 4\cos(\pi t/2 - \pi/3)$

3.3 $x[n] = \cos(2\pi(n - 8)/24)$
$y[n] = x[3n - 4] = -\cos(\pi n/4)$

3.4 $x[n] = \cos(2\pi(n - 5)/24)$
$y[n] = x[2n + 5] = \cos(\pi n/6)$

3.5



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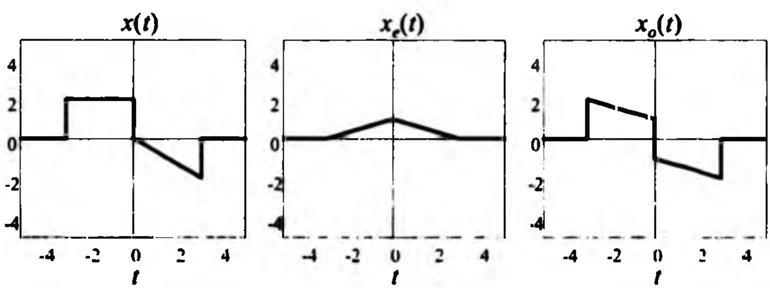

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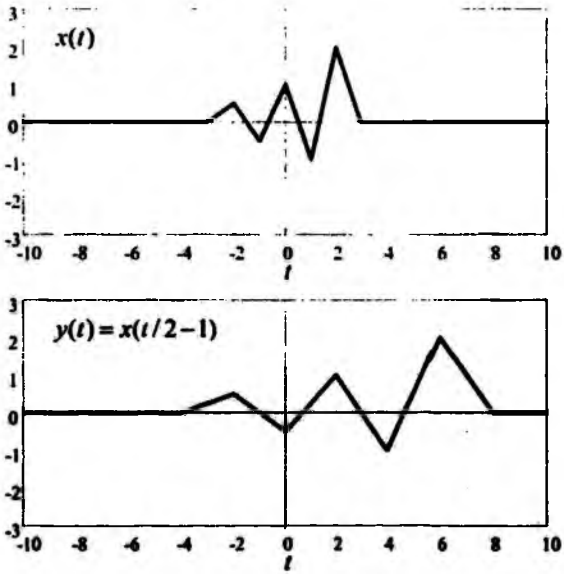

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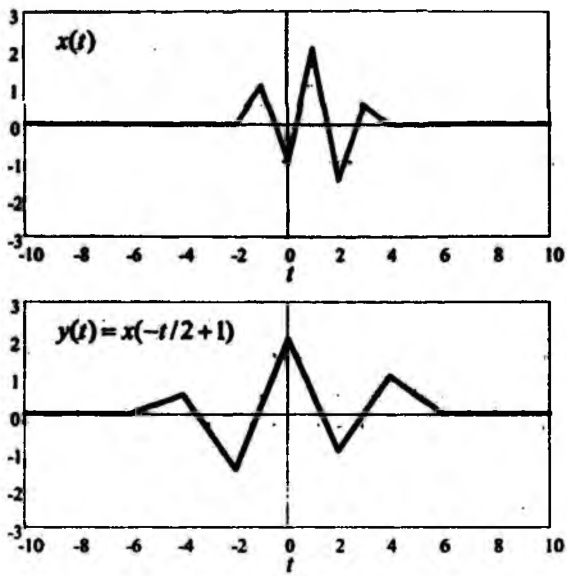

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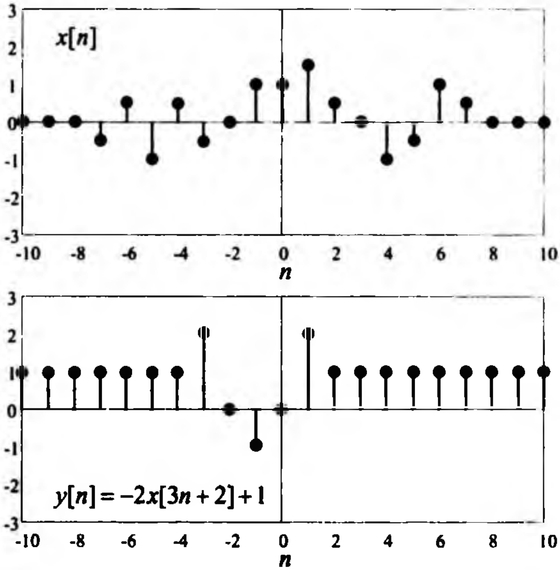

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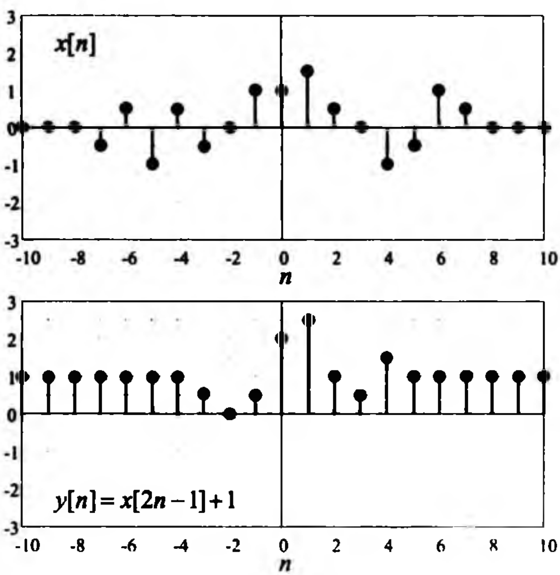
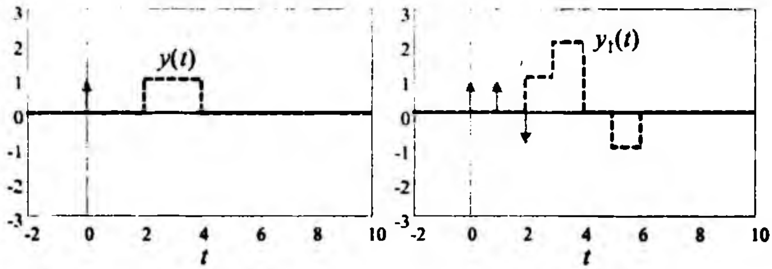


CHAPTER 4

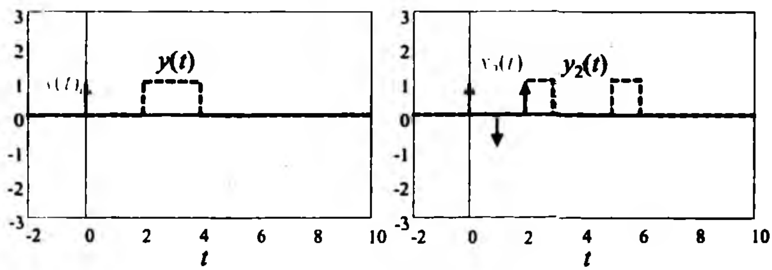
4.3 a

4.4 b

4.5



4.6



4.7 $y[n] = x[n-1] + x[n-2]$

4.8 $y[n] = x[n] + x[n-2]$

4.9 $y[n] = x[n-1] - 2x[n-3] + x[n-4]$

4.10 $y[n] = 2x[n] - 3x[n-1] + x[n-2]$

CHAPTER 5

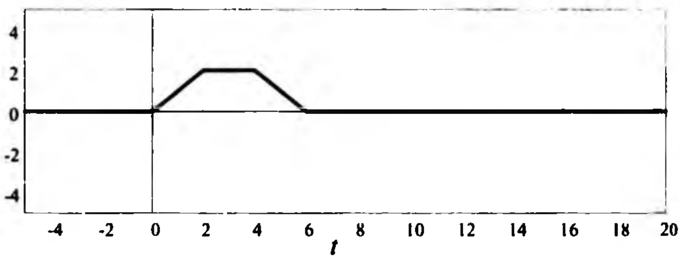
5.1 $x(t) * h(t) = t^3 u(t)/6$

5.2 $x(t) * h(t) = t^3 u(t)/3$

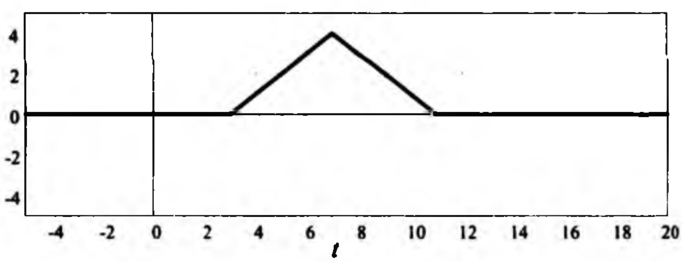
5.3 $x(t) * h(t) = \sin(\pi t)\, u(t)/\pi$

5.4 $x(t) * h(t) = [1 - \cos(\pi t)]\, u(t)\, u(2 - t)/\pi$

5.5 $x(t) * h(t)$



5.6 $x(t) * h(t)$



5.7 d

5.8 b

CHAPTER 6

6.1 $x[n] * h[n] = n(n+1)(2n+1)\,u[n]/6$

6.2 $x[n] * h[n] = (3^{n+1} - 2^{n+1})\,u[n]$

6.3 $x[n] * h[n] = (a^{n+1} - b^{n+1})\,u[n]/(a-b)$

6.4 $x[n] * h[n] = c^n(n+1)\,u[n]$

6.5 $x[n] * h[n] = \delta[n] + 2\delta[n-1] - 3\delta[n-2] + 2\delta[n-3] - 2\delta[n-4]$

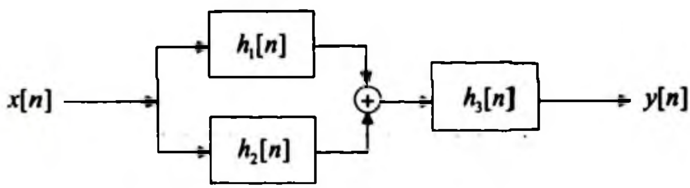
6.6 $x[n] * h[n] = \delta[n] - 2\delta[n-1] - 3\delta[n-2] + 2\delta[n-3] + 2\delta[n-4]$

6.7 $h[n] = 2\delta[n] - 3\delta[n-1]$

6.8 $h[n] = 2\delta[n-1] + \delta[n-2] - \delta[n-3] + \delta[n-4]$

6.9 $y[n] = x[n] * h_1[n] * (h_2[n] * h_3[n] + h_4[n])$

6.10

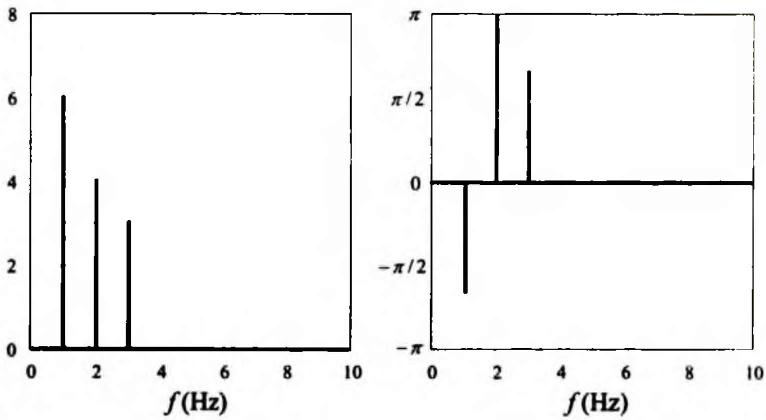


CHAPTER 7

7.1 $\Omega = \pi/3, \quad a_0 = 2, \quad a_2 = 1, \quad \text{and} \quad b_5 = 4$

7.2 $\Omega = 4\pi, \quad a_m = 0, \quad \text{and} \quad b_m = 2\cos^2(m\pi)/m$

7.5



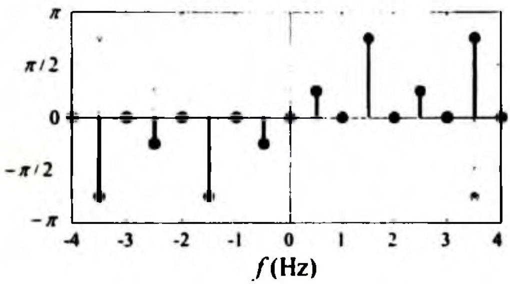
7.6 $X[-3] = j/8$
$X[-2] = 1/4$
$X[-1] = -3j/8$
$X[0] = 3/2$
$X[1] = 3j/8$
$X[2] = 1/4$
$X[3] = -j/8$

7.9 d

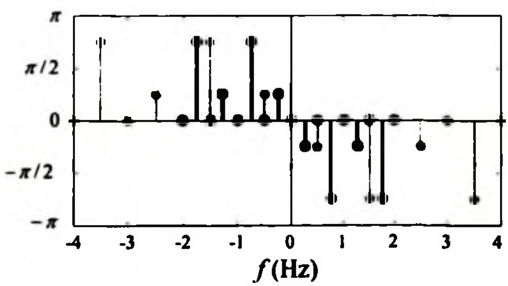
7.10 c

CHAPTER 8

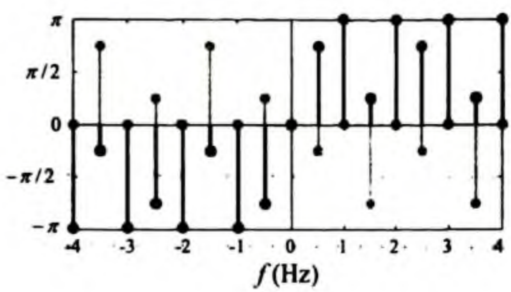
8.1 $\psi_m = -\theta_m$



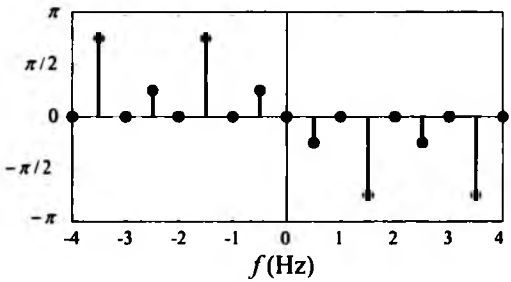
8.2 time expansion = frequency compression



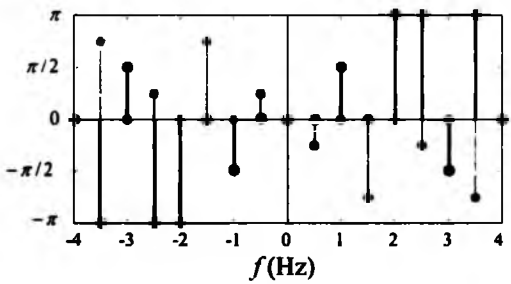
8.3 $\psi_m = \theta_m \pm \pi$ for $m \neq 0$



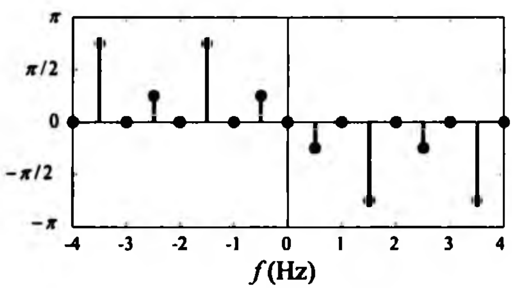
8.4 $\psi_m = \theta_m$



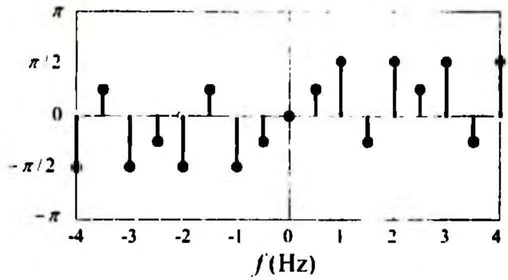
8.5 $\psi_m = \theta_m + \pi m/4$



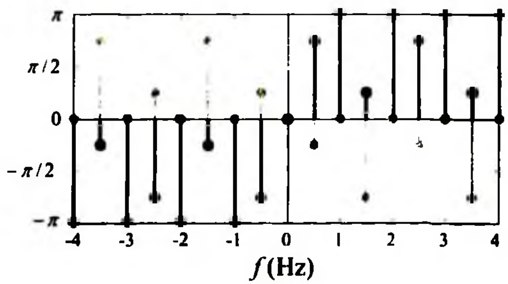
8.6 $\psi_m = \theta_m - 2\pi m$



8.7 $\psi_m = \theta_m + \pi/2$ $(m > 0)$, $\psi_m = \theta_m - \pi/2$ $(m < 0)$



8.8 $\psi_m = \theta_m + \pi$ $(m > 0)$, $\psi_m = \theta_m - \pi$ $(m < 0)$



8.9 $P = 58$

8.10 $P = 8.5$

CHAPTER 9

9.1 a

9.2 d

9.3 $Y(\omega) = X(\omega) - X(-\omega) = 2/(j\omega) + 2j\sin\omega/\omega^2$

9.4 $Y(\omega) = X(\omega) + X(-\omega) = 2/\omega^2 - 2\cos\omega/\omega^2$

9.5 $y(t) = x(t-2)$
$Y(\omega) = e^{-j2\omega}X(\omega) = 2e^{-j2\omega}\sin\omega/\omega = (e^{-j\omega} - e^{-j3\omega})/(j\omega)$

9.6 $y'(t) = \delta(t-1) - \delta(t-3)$
$j\omega Y(\omega) = e^{-j\omega} - e^{-j3\omega}$
$Y(\omega) = (e^{-j\omega} - e^{-j3\omega})/(j\omega)$

9.7 $Y(\omega) = \int_1^3 e^{-j\omega t}dt = (e^{-j\omega} - e^{-j3\omega})/(j\omega)$

9.8 $y(t) = -x(t-1)$
$Y(\omega) = -e^{-j\omega}X(\omega) = -2e^{-j\omega}\sin\omega/\omega = (e^{-j2\omega} - 1)/(j\omega)$

9.9 $y'(t) = \delta(t-2) - \delta(t)$
$j\omega Y(\omega) = e^{-j2\omega} - 1$
$Y(\omega) = (e^{-j2\omega} - 1)/(j\omega)$

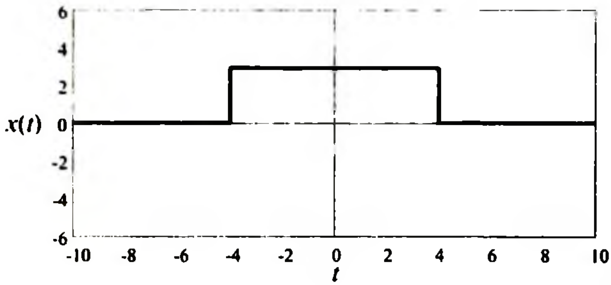
9.10 $Y(\omega) = -\int_0^2 e^{-j\omega t}dt = (e^{-j2\omega} - 1)/(j\omega)$

CHAPTER 10

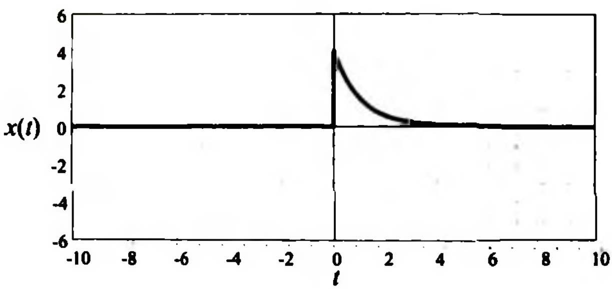
10.1 $h(t) = 4\delta(t) - 8e^{-2t}u(t)$

10.2 $h(t) = 2\delta(t) - 6e^{-3t}u(t)$

10.3



10.4



10.5 $y(t) = (e^{-2t} - e^{-3t})\,u(t)$

10.6 $y(t) = (e^{-t} - e^{-3t})\,u(t)/2$

10.7 $v_C(t) = (8e^{-t/2} - 8e^{-t})\,u(t)$ V

10.8 $i_C(t) = (16e^{-2t} - 8e^{-t})\,u(t)$ A

CHAPTER 11

11.1 $x[n] = \delta[n] + 3\delta[n-2] - 2\delta[n-3] + \delta[n-4]$

11.2 $x[n] = 2\delta[n] - \delta[n-2] + 3\delta[n-3] - 2\delta[n-4]$

11.3 $\text{Re}[X(\omega)] = 1 + 2\cos(\omega) - 3\cos(2\omega)$
$\text{Im}[X(\omega)] = -2\sin(\omega) + 3\sin(2\omega)$

11.4 $\text{Re}[X(\omega)] = 1 - 3\cos(\omega) + 2\cos(2\omega)$
$\text{Im}[X(\omega)] = 3\sin(\omega) - 2\sin(2\omega)$

11.5 $X[m] = \{1,\ 1,\ 5,\ 1\}$

11.6 $X[m] = \{0,\ 4j,\ 0,\ -4j\}$

11.7 $x[n] = \{2,\ -1,\ 1,\ -1\}$

11.8 $x[n] = \{0,\ -2,\ 0,\ 2\}$

11.9 $X[2] = 2$

11.10 $X[3] = j$

CHAPTER 12

12.1 $f_{\min} = 0.05$ Hz and $f_{\max} = 100$ Hz

12.2 $f_{\min} = 0.10$ Hz and $f_{\max} = 250$ Hz

12.3 $\Delta t = 10^{-7}$ s and $t_{\max} = 5 \times 10^{-5}$ s

12.4 $\Delta t = 5 \times 10^{-7}$ s and $t_{\max} = 2 \times 10^{-4}$ s

APPENDIX C

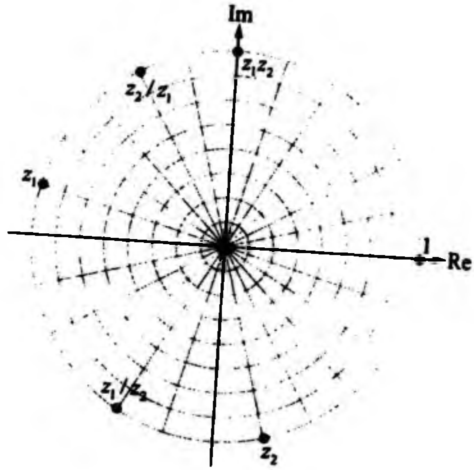
C.3 $z = e^{-j\pi/4}/\sqrt{2}$

C.4 $z = e^{-j\pi/3}$

C.5 $z^{2020} = -z$

C.6 $z^{2022} = -1$

C.7



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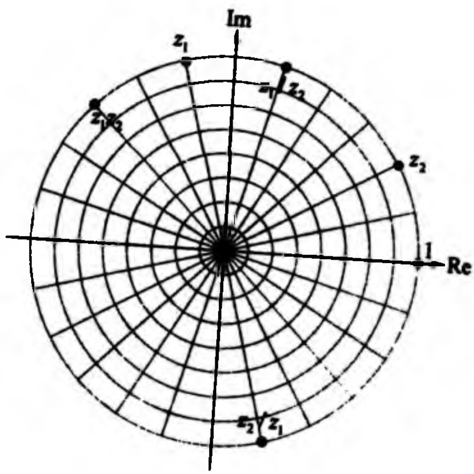

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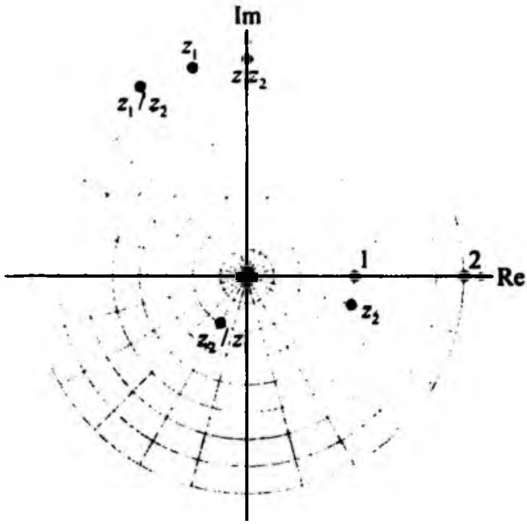

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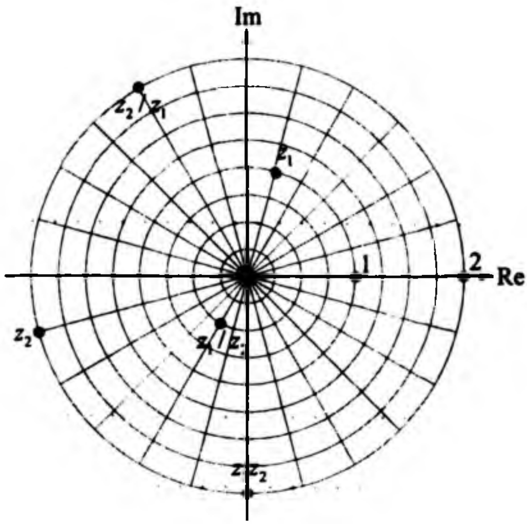

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